Project: **Time series analysis of elasto-plastic bifurcations based on extremely short observation times**

Principal researcher: Prof. Dr. H. Kantz, Max-Planck-Institute for the Physics of Complex Systems, Dresden

---

Frequency-domain conditions for observers that determine the output convergence of a variational inequality

Volker Reitmann

Max-Planck-Institute for
the Physics of Complex Systems, Dresden

SPP 1114, 2nd ANNUAL MEETING, Teistungenburg, February 26 - 28, 2003

1 Basic notation

Suppose that Y_0 is a real Hilbert space with $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ as scalar product resp. norm. Suppose also that $A : \mathcal{D}(A) \subset Y_0 \to Y_0$ is an unbounded densely defined linear operator. The Hilbert space Y_1 is defined as $\mathcal{D}(A)$ equipped with the scalar product

$$(y,\eta)_1 := ((\beta I - A)y,\, (\beta I - A)\eta)_0 \;, \quad y,\eta \in \mathcal{D}(A)\,, \tag{1.1}$$

where $\beta \in \rho(A)$ (the resolvent set of A) is an arbitrary but fixed number.
The Hilbert space Y_{-1} is by definition the completion of Y_0 with respect to the norm $\|z\|_{-1} := \|(\beta I - A)^{-1}z\|_0$. Thus we have the dense and continuous imbeddings

$$Y_1 \subset Y_0 \subset Y_{-1} \tag{1.2}$$

which is called *Hilbert space rigging structure*. In this triple, Y_0 is the *pivot space*, Y_1 is the *interpolation space*, and Y_{-1} is the *extrapolation space* (Triebel [14]).
The *duality product* $(\cdot,\cdot)_{-1,1}$ on $Y_{-1}\times Y_1$ is the unique extension by continuity of the scalar product $(\cdot,\cdot)_0$ defined on $Y_0 \times Y_1$.
If $T > 0$ is an arbitrary number we define the norm for Bochner measurable functions in $L^2(0,T;Y_j)\;,\; j = 1,0,-1$, through

$$\|y(\cdot)\|_{2,j} := \left(\int_0^T \|y(t)\|_j^2\, dt\right)^{1/2} . \tag{1.3}$$

Let $\mathcal{W}_T$ be the space of functions $y(\cdot) \in L^2(0,T;Y_1)$ for which $\dot{y}(\cdot) \in L^2(0,T;Y_{-1})$ equipped with the norm

$$\|y(\cdot)\|_{\mathcal{W}_T} := (\|y(\cdot)\|_{2,1}^2 + \|\dot{y}(\cdot)\|_{2,-1}^2)^{1/2} . \tag{1.4}$$

2 Evolutionary variational inequalities

Suppose $Y_1 \subset Y_0 \subset Y_{-1}$ is a real Hilbert space rigging structure with $A \in \mathcal{L}(Y_1, Y_{-1})$.
Assume that Ξ and W are two real Hilbert spaces with scalar products $(\cdot,\cdot)_\Xi\,,\,(\cdot,\cdot)_W$ and norms $\|\cdot\|_\Xi\;,\;\|\cdot\|_W$, respectively.
Introduce the linear continuous operators

$$B : \Xi \to Y_{-1}\,, \quad C : Y_{-1} \to \Xi \tag{2.1}$$

and define the set-valued map

$$\varphi : \mathbb{R}_+ \times W \to 2^{\Xi} \tag{2.2}$$

and the map

$$\psi : Y_1 \to \mathbb{R}_+ \cup \{+\infty\} \;. \tag{2.3}$$

Consider the *evolutionary variational inequality* (Duvant, Lions [4])

$$(\dot{y} - Ay - B\xi, \eta - y)_{-1,1} + \psi(\eta) - \psi(y) \ge 0\,, \quad \forall \eta \in Y_1\;, \tag{2.4}$$

$$w(t) = Cy(t)\,, \quad \xi(t) \in \varphi(t, w(t))\,, \quad y(0) = y_0 \in Y_0\,. \tag{2.5}$$

Note that in applications φ is a *material law nonlinearity*, ψ is a *contact-type* or *friction functional* and $w(t) = Cy(t)$ is the *output* of the inequality.
In the contact free case when $\psi \equiv 0$ the evolutionary variational inequality (2.4) $-$ (2.5) is equivalent to an *evolution equation* with a set-valued nonlinearity φ given by

$$\begin{aligned} \dot y &= Ay + B\xi \quad \text{in} \quad Y_{-1}\,, && (2.4)' \\ w(t) &= Cy(t)\,, \xi(t) \in \varphi(t, w(t))\,, y(0) = y_0 \in Y_0\,. && (2.5)' \end{aligned}$$

A function $y(\cdot) \in \mathcal{W}_T$ is said to be a *solution* of (2.4), (2.5) on $(0,T)$ if there exists a function $\xi(\cdot) \in L^2(0,T;\Xi)$ such that for a.a. $t \in (0,T)$ the inequality (2.4), (2.5) is satisfied and $\int_0^T \psi(y(t))dt < +\infty$. The pair $\{y(\cdot), \xi(\cdot)\}$ is called a *response* of (2.4), (2.5); $\xi(\cdot)$ is an associated *selection.*

Remark 2.1 (Lions [12]) $\mathcal{W}_T$ can be continuously imbedded into the space C_T of continuous mappings $[0,T] \to Y_0$ equipped with the norm $\|y(\cdot)\|_{C_T} := \sup_{t\in[0,T]} \|y(t)\|_0$. Thus, every function from $\mathcal{W}_T$, properly altered by some set of measure zero, is a continuous function $[0,T] \to X_0$ and $\|y(\cdot)\|_{C_T} \leq \text{ const}\|y(\cdot)\|_{\mathcal{W}_T}$.
It follows that the value of a function $y(\cdot) \in \mathcal{W}_T$ at the point $t = 0$ has a meaning and the initial condition (2.5) is well-posed. □

Suppose that F and G are two quadratic forms on $Y_1 \times \Xi$. The *class* $\mathcal{N}(F,G)$ *of nonlinearities* for (2.4), (2.5) consists of all maps (2.2) such that the following two conditions are satisfied:

a) For any $T > 0$ and any two pairs of functions $y_1(\cdot), y_2(\cdot) \in L^2(0,T;Y_1)$ and $\xi_1(\cdot), \xi_2(\cdot) \in L^2(0,T;\Xi)$ with

$$\xi_i(t) \in \varphi(t, Cy_i(t))\,, \text{ for a.a. } t \in [0,T] \tag{2.6}$$

it follows that

$$F(y_1(t) - y_2(t), \xi_1(t) - \xi_2(t)) \geq 0\,, \text{ a.a. } t \in [0,T]\,. \tag{2.7}$$

b) For any $T > 0$ and any two pairs of functions

$$y_1(\cdot), y_2(\cdot) \in L^2(0,T;Y_1) \text{ and } \xi_1(\cdot), \xi_2(\cdot) \in L^2(0,T;\Xi)$$

satisfying (2.6) there exist a continuous function $\Phi : W \to \mathbb{R}$ (*generalized potential*) and a number $\gamma > 0$ (both may be depending on the given functions) such that

$$\int_s^t G\big(y_1(\tau) - y_2(\tau), \xi_1(\tau) - \xi_2(\tau)\big)\, d\tau \geq \frac{1}{2}\Big[\Phi(Cy_1(t) - Cy_2(t)) - \Phi(Cy_1(s) - Cy_2(s))\Big]$$

$$+ \lambda \int_s^t \Phi(Cy_1(\tau) - Cy_2(\tau))d\tau \quad \text{for all} \quad 0 \leq s < t \leq T \tag{2.8}$$

and

$$\Phi(Cy_1(t) - Cy_2(t)) \geq \gamma \|Cy_1(t) - Cy_2(t)\|_W^2 \,, \quad \text{for a.a. } t \in [0,T] \,. \tag{2.9}$$

(A1) For fixed linear operators A, B, C, fixed function (2.3) and arbitrary $y_0 \in Y_0, T > 0$ and $\varphi \in \mathcal{N}(F,G)$ there exists a response $\{y(\cdot), \xi(\cdot)\}$ of (2.4), (2.5).

Example 2.1 Suppose that $\Omega \subset \mathbb{R}^n$ is a domain with smooth boundary $\Gamma = \partial\,\Omega$, $h : \Gamma \to \mathbb{R}$ is a given scalar function ("outer pressure") and $u(x,t)$ (" inner pressure") is a solution of

$$\frac{\partial u}{\partial t} = \triangle u \quad \text{in} \quad \Omega \times \mathbb{R}_+ \tag{2.10}$$

subject to the boundary conditions

$$u = h \quad \text{on } \Gamma \times \mathbb{R}_+ \quad \Rightarrow \quad \frac{\partial u}{\partial n} \geq 0 \,, \tag{2.11}$$

$$u > h \quad \text{on } \Gamma \times \mathbb{R}_+ \quad \Rightarrow \quad \frac{\partial u}{\partial n} = 0 \tag{2.12}$$

and the initial condition

$$u(\cdot, 0) = u_0 \,. \tag{2.13}$$

The system (2.10) – (2.13) describes the transfer problem of fluid acrossing a semi-permeable membrane (Lions [12]).
Instead of (2.11) – (2.12) we consider the (nonlinear) boundary condition

$$\frac{\partial u}{\partial n} \geq g \quad \text{on} \quad \Gamma \times \mathbb{R}_+ \,, \tag{2.14}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given function.
In order to get a representation of (2.10) – (2.14) in the form of a variational inequality (2.4), (2.5) we introduce the spaces

$$\begin{aligned} Y_0 &:= L^2(\Omega) \,, \\ Y_1 &:= W^{1,2}(\Omega) = \{v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega) \,, \ i = 1, 2, \ldots, n\} \text{ and} \\ \Xi &:= W^{-1/2,2}(\partial\Omega) \,. \end{aligned}$$

An operator $A \in \mathcal{L}(Y_1, Y_{-1})$ is defined by

$$(Au, v)_{-1,1} = -\int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \,, \quad \forall u, v \in Y_1 \,. \tag{2.15}$$

The operator $B \in \mathcal{L}(\Xi, Y_{-1})$ is given by

$$(B\xi, y)_{-1,1} = -\int_{\partial\Omega} \xi y dS\,, \quad \forall \xi \in \Xi\,, \quad \forall y \in Y_1\,, \tag{2.16}$$

the nonlinear map $\varphi : Y_1 \to \Xi$ is given by

$$\varphi(y(x)) := g\,(y)(x) \quad \text{on} \quad \Gamma\,, \tag{2.17}$$

and the "contact functional" $\psi : Y_1 \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\psi(\eta) := \begin{cases} 0 & \text{if } \eta(x) \geq h(x) \text{ on } \Gamma\,, \\ +\infty & \text{in other cases.} \end{cases} \tag{2.18}$$

Thus the transfer problem of fluid (2.10) – (2.14) can be considered as evolutionary variational inequality

$$(\dot{y} - Ay - B\xi\,,\, \eta - y)_{-1,1} + \psi(\eta) - \psi(y) \geq 0\,, \quad \forall \eta \in Y_1\,, \tag{2.19}$$

$$\xi(t) = \varphi(y(t))\,, \quad y(0) = y_0 \in Y_0\,. \tag{2.20}$$

Let us describe the class $\mathcal{N}(F, G)$ for (2.19), (2.20). We assume that the nonlinearity φ from (2.17) has the following two properties:

(H1) $\exists \mu_0 > 0 \quad \forall y_1, y_2 \in Y_1 \quad :$

$$0 \leq (B\varphi(y_1) - B\varphi(y_2)\,,\, y_1 - y_2)_{-1,1} \leq \mu_0 \|y_1 - y_2\|_1^2\,. \tag{2.21}$$

(H2) There exist a Fréchet differentiable map $\Phi : Y_0 \to \mathbb{R}$ and a number $\lambda > 0$ such that with the Fréchet derivative $\Phi' \in \mathcal{L}(Y_0, \mathbb{R})$ the inequality

$$(\varphi(y), \eta)_1 \geq \Phi'(y)\eta + \lambda\Phi(\eta)\,, \quad \forall \eta \in Y_1 \tag{2.22}$$

is satisfied.

It is clear that (2.21) and (2.22) can be considered as a monotonicity condition and a potential-type condition, respectively. Using (2.21) we can introduce the quadratic form

$$F(y, \xi) := \mu_0 \|y\|_1^2 - (B\xi, y)_{-1,1}\,, \quad (y, \xi) \in Y_1 \times \Xi\,, \tag{2.23}$$

which satisfies (2.7). The inequality (2.22) can be used to define the quadratic form

$$G(y, \xi) := (G_1 Ay, \xi)_\Xi + (G_2 B\xi, \xi)_\Xi \quad \text{on} \quad Y_1 \times \Xi \tag{2.24}$$

with $G_i : Y_{-1} \to \Xi \quad (i = 1, 2)$. It is easy to see that the form G from (2.24) and the generalized potential Φ from (2.22) satisfy the inequality (2.8) . □

3 Observations that completely determine the asymptotic behaviour of an output

Assume that Z is a real Hilbert space with norm $\|\cdot\|_Z$ and

$$D : Y_1 \to Z\,, \quad E : \Xi \to Z \tag{3.1}$$

are linear bounded operators which are called *observation operators* with respect to the *observation space* Z. If $\{y(\cdot), \xi(\cdot)\}$ is an arbitrary response of (2.4), (2.5) then

$$z(\cdot) = Dy(\cdot) + E\,\xi(\cdot) : \mathbb{R}_+ \to Z \tag{3.2}$$

with $z(\cdot) \in L^2_{loc}(\mathbb{R}_+, Z)$ is an *observation* (*measurement* or *time series*) of the response. Suppose that $\alpha \geq 0$ is a number. The observation (3.2) is *determining* for the *output α-convergence* in (2.4), (2.5) if for any two responses $\{y_1(\cdot), \xi_1(\cdot)\}$ and $\{y_2(\cdot), \xi_2(\cdot)\}$ of (2.4), (2.5) from

$$\int_t^{t+1} \|D(y_1(\tau) - y_2(\tau)) + E(\xi_1(\tau) - \xi_2(\tau))\|_Z^2 d\tau \to 0 \quad \text{for } t \to +\infty \tag{3.3}$$

it follows that

$$\limsup_{t\to+\infty} \|C(y_1(t) - y_2(t))\|_W \leq \alpha \tag{3.4}$$

(Foias, Prodi [5], Ladyzhenskaya [10], Foias, Temam [6], Chuesov [2, 3]). Inverse problems for variational inequalities (parameter identification) are considered by Hoffmann, Sprekels [7], Maksimov [13] and other authors.

Example 3.1 Suppose that $\{y(\cdot), \xi(\cdot)\}$ is an arbitrary response of the membrane problem (2.19) – (2.20). The following operators can be used as observation operators (with $E = 0$, $y(\cdot) = u(x, \cdot)$ an arbitrary solution) :

a) $$Z = \mathbb{R}, \quad z(t) = Dy(t) = \int_\Omega u(x,t)dx\,, \quad t > 0\,; \tag{3.5}$$

b) $$Z = \mathbb{R}, \quad z(t) = u(x_0, t)\,, \quad t > 0\,, \quad x_0 \in \Omega \text{ fixed;} \tag{3.6}$$

c) $$Z = \mathbb{R}^N\,, \quad \bar{\Omega} = \bigcup_{j=1}^N \bar{\Omega}_j\,, \quad \Omega_i \cap \Omega_j = \emptyset\,, \quad i \neq j\,,$$

$$\lambda_j(\cdot) \in L^\infty(\Omega_j)\,, \quad \lambda_j(x) \geq 0 \quad \text{in } \Omega_j\,, \quad j = 1, \dots, N\,,$$

$$z(t) = \left(\int_{\Omega_1} \lambda_1(x)u(x,t)dx, \dots, \int_{\Omega_N} \lambda_N(x)u(x,t)dx\right), \quad t > 0\,. \tag{3.7}$$

Then, for example, z from b) is determining for the α-convergence of the full output in (2.19) – (2.20) if for arbitrary solutions $u_1(x,t)$ and $u_2(x,t)$ it follows from

$$\int_t^{t+1} (u_1(x_0,\tau) - u_2(x_0,\tau))^2 d\tau \to 0 \quad \text{for } t \to +\infty$$

that

$$\limsup_{t\to+\infty} \|u_1 - u_0\| = \limsup_{t\to+\infty} \left(\int_\Omega (u_1(x,t) - u_2(x,t))^2 dx\right)^{1/2} \leq \alpha.$$

□

4 Lyapunov-type approach for the construction of determining observations

Suppose that two arbitrary responses $\{y_1(\cdot), \xi_1(\cdot)\}$ and $\{y_2(\cdot), \xi_2(\cdot)\}$ of (2.4), (2.5) are given. We want to find observation operators D and E satisfying (3.1) and an *energy-type* operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ such that the following conditions are satisfied.

a) $V_1(y) := \frac{1}{2}\,(y, Py)_0 \geq 0 \quad , \quad \forall y \in Y_0;$

b) $V(y_1(t) - y_2(t)) := V_1(y_1(t) - y_2(t)) + \frac{1}{2}\,\Phi\,(Cy_1(t) - Cy_2(t))$
$$\geq \text{const.} \|Cy_1(t) - Cy_2(t)\|_W^2$$
for a.a. $t \geq 0$, where Φ is a generalized potential for the given pairs of responses;

c) There exists a number $\lambda > 0$ such that the functions $m(t) := V(y_1(t) - y_2(t))$, $g(t) := \|D(y_1(t) - y_2(t) + E(\xi_1(t) - \xi_2(t))\|_Z^2$ and a function $p\,(\cdot) \in L^1(\mathbb{R}_+; \mathbb{R}_+)$ satisfy the inequality
$$\dot{m}(t) + 2\,\lambda m(t) + p\,(t) \leq g(t)\,, \quad \text{a.a. } t \geq 0. \tag{4.1}$$

Since $\int\limits_0^\infty p(\tau)d\tau \geq -c_1$ with some $0 < c_1 < +\infty$ and $\lim\limits_{t\to+\infty} \int\limits_t^{t+1} g(\tau)d\tau = 0$, we get from (4.1) that $\limsup\limits_{t\to+\infty} m(t) \leq \frac{c_1}{2\lambda}$, i.e., $\limsup\limits_{t\to+\infty} V_1(y_1(t) - y_c(t)) + \frac{1}{2}\Phi(Cy_1(t) - Cy_2(t)) \leq \frac{c_1}{2\lambda}$ and, using (2.9) and the property a)
$$\limsup_{t\to+\infty} \|Cy_1(t) - Cy_2(t)\|_W \leq \left(\frac{c_1}{\gamma\lambda}\right)^{1/2} =: \alpha \tag{4.2}$$

It follows that the observation $z(\cdot) = Dy(\cdot) + E\xi(\cdot)$ is determining for the output α-convergence with α from (4.2). If the contact functional satisfies $\psi \equiv 0$, i.e. the evolution inequality (2.4), (2.5) is equivalent to the evolution *equation* $(2.4)', (2.5)'$, the considered observation is determining for the output convergence in the usual sense.

5 Frequency-domain methods for the construction of determining observers

Our goal is to find effective conditions for the existence of Lyapunov-type functions V satisfying a) – c) in the Section 4. A general approach consists in using the *Frequency Theorem* which is also called *Kalman-Yakubovich-Popov Lemma* (KYP Lemma [1, 11]). Let us state the assumptions for this theorem.

(A2) There exists a number $\lambda > 0$ such that for any $T > 0$ and any $f \in L^2(0, T; Y_{-1})$ the problem
$$\dot{y} = (A + \lambda I)y + f(t)\,,\; y(0) = y_0 \tag{5.1}$$

is well-posed, i.e., for arbitrary $y_0 \in Y_0, f(\cdot) \in L^2(0,T;Y_{-1})$ there exists an unique solution $y(\cdot) \in \mathcal{W}_T$ satisfying (5.1) in the sense that

$$(\dot{y},\eta)_{-1,1} = ((A+\lambda I)y,\eta)_{-1,1} + (f(t)\ ,\eta)_{-1,1}\ , \quad \forall \eta \in Y_1,\ \text{a.a. } t \in [0,T]\,, \tag{5.2}$$

and depending continuously on the initial data, i.e.,

$$\|y(\cdot)\|^2_{\mathcal{W}_T} \le c_1\|y_0\|^2_0 + c_2\|f(\cdot)\|^2_{2,-1}\ , \tag{5.3}$$

where $c_1 > 0$ and $c_2 > 0$ are some constants. Furthermore, any solution of

$$\dot{y} = (A+\lambda I)y, \quad y(0) = y_0 \tag{5.4}$$

is exponentially decreasing for $t \to +\infty$, i.e., there exist constants $c_3 > 0$ and $\varepsilon > 0$ such that

$$\|y(t)\|_0 \le c_3 e^{-\varepsilon t}\|y_0\|_0\ ,\ t > 0\ . \tag{5.5}$$

(A3) There exists a number $\lambda > 0$ such that the operator $A + \lambda I \in \mathcal{L}(Y_1, Y_{-1})$ is *regular*, i.e., for any $T > 0, y_0 \in Y_1, z_T \in Y_1$ and $f \in L^2(0,T;Y_0)$ the solutions of the *direct problem*

$$\dot{y} = (A+\lambda I)y + f(t), \quad y(0) = y_0, \quad \text{a.a. } t \in [0,T]\,, \tag{5.6}$$

and of the *dual problem*

$$\dot{z} = -(A+\lambda I)^* z + f(t), \quad z(0) = z_T, \quad \text{a.a. } t \in [0,T]\,, \tag{5.7}$$

are strongly continuous in t in the norm of Y_1.

In the next assumption which is called *frequency-domain condition* it is necessary to consider the *complexification* of spaces and linear operators under consideration.
The elements of the complexification Y_0^c of the real Hilbert space Y_0 can be written as $x+iy$ with $x, y \in Y_0$, and the inner product of Y_0^c will be denoted by $(\cdot,\cdot)_{Y_0^c}$. The complexification of the other spaces are defined in a similar way. For the linear operator $A : Y_1 \to Y_{-1}$ we denote by A^c the linear operator $A^c : Y_1^c \to Y_{-1}^c$ defined by $A^c(x+iy) = Ax + iAy$. Again, the complexification of the other linear operators which will appear below, is defined in a similar way.
Consider now the complexification of the quadratic form F (similarly of G). Suppose that

$$F(y,\xi) = (F_1 y, y)_{-1,1} + 2(F_2 y, \xi)_\Xi + (F_3\xi,\xi)_\Xi \tag{5.8}$$

for $(y,\xi) \in Y_1 \times \Xi$, where $F_1 = F_1^* \in \mathcal{L}(Y_1, Y_{-1}), F_2 \in \mathcal{L}(Y_1,\Xi)$ and $F_3 = F_3^* \in \mathcal{L}(\Xi,\Xi)$.
The complexification of the quadratic form (5.8) is the Hermitian form F^c defined on $Y_1^c \times \Xi^c$ by

$$F^c(y,\xi) = (F_1^c y, y)_{Y_{-1}^c, Y_1^c} + 2\mathrm{Re}\,(F_2^c y,\xi)_{\Xi^c} + (F_3^c\xi,\xi)_{\Xi^c}\ . \tag{5.9}$$

(A4) (Frequency-domain condition)
There exist numbers $\lambda > 0$ and $\delta > 0$ such that the following two properties hold:

a) $$F^c(y,\xi) + G^c(y,\xi) - \delta \|D^c y + E^c \xi\|^2_{Z^c} \le 0 \tag{5.10}$$

for all such $(y,\xi) \in Y_1^c \times \Xi^c$ for that there exists an

$$\omega \in \mathbb{R} \quad \text{with} \quad i\omega\, y = (A^c + \lambda I^c) y + B^c y\,;$$

b) The functional

$$J(y(\cdot),\xi(\cdot)) := \int_0^\infty [F^c(y(\tau),\xi(\tau)) + G^c(y(\tau),\xi(\tau)) - \delta \|D^c y(\tau) + E^c \xi(\tau)\|^2_{Z^c}] d\tau \tag{5.11}$$

is bounded from above on the set

$$\mathcal{M}_{y_0} := \Big\{ y(\cdot),\xi(\cdot) : \dot y = (A^c + \lambda I^c) y + B^c \xi, \\ y(0) = y_0\,,\ y(\cdot) \in \mathcal{W}^c_\infty\,,\ \xi(\cdot) \in L^2(0,\infty;\Xi^c) \Big\}$$

for any $y_0 \in Y_0^c$.

Theorem 5.1 *Suppose that there exist numbers $\lambda > 0$ and $\delta > 0$ such that the assumptions* **(A1)** - **(A4)** *are satisfied for (2.2) - (2.5) with $\varphi \in \mathcal{N}(F,G)$ and an observation given by (3.2). Then the observation (3.2) is determining for the output α-convergence in (2.4), (2.5), where α is defined by (4.2).*

Idea of the proof: We try to find an operator $P = P^* \in \mathcal{L}(Y_{-1},Y_0) \cap \mathcal{L}(Y_0,Y_1)$ with $(y,Py)_0 \ge 0$, $\forall y \in Y_0$, and numbers $\lambda > 0, \delta > 0$ such that for any two responses $\{y_1(\cdot),\xi_1(\cdot)\}$ and $\{y_2(\cdot),\xi_2(\cdot)\}$ of (2.4), (2.5) and their associated generalized potential Φ from condition (2.8) the integrated inequality (4.1) is true on any time interval $0 < s < t$, i.e.,

$$m(t) - m(s) + 2\lambda \int_s^t m(\tau) d\tau + \int_s^t p(\tau) d\tau \le \int_s^t g(\tau) d\tau\,. \tag{5.12}$$

In (5.12) we have introduced the functions

$$m(t) := \frac{1}{2}\Big(y_1(t) - y_2(t), P\big(y_1(t) - y_2(t)\big)\Big)_0 + \frac{1}{2}\Phi(y_1(t) - y_2(t))\,, \tag{5.13}$$

$$p(t) := \psi(y_1(t)) - \psi\Big(y_1(t) - P\big(y_2(t) - y_1(t)\big)\Big) - \psi\Big(y_2(t) + P\big(y_1(t) - y_2(t)\big)\Big) + \psi(y_2(t))\,, \tag{5.14}$$

and

$$g(t) := -\,\delta \|D(y_1(t) - y_2(t)) + E(\xi_1(t) - \xi_2(t))\|_Z^2\,. \tag{5.15}$$

In order to guarantee the inequality (5.12) we choose an operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and numbers $\lambda > 0, \delta > 0$ such that

$$(-(A+\lambda I)v - B\zeta, Pv)_{-1,1} \geq F(v,\zeta) + G(v,\zeta) -\ \delta\|Dv + E\zeta\|_Z^2\,, \quad \forall y \in Y_1\,, \quad \forall \zeta \in \Xi\,. \tag{5.16}$$

The existence of such a P with $(y, Py)_0 \geq 0\,, \quad \forall y \in Y_0$, follows due to the assumptions **(A2)** – **(A4)** from the infinite-dimensional version of the Kalman-Yakubovich-Popov Lemma (Frequency Theorem [1, 11]). From (2.4), (2.5) it follows with $v(t) := y_1(t) - y_2(t)$ and $\zeta(t) := \xi_1(t) - \xi_2(t)$ that

$$(\dot v(t), Pv(t))_{-1,1} + \lambda (v(t), Pv(t))_0 - ((A+\lambda I)v(t) +\ B\zeta(t), Pv(t))_{-1,1} + p(t) \leq 0\,, \quad \text{a.a. } t > 0\,. \tag{5.17}$$

Using the estimate (5.16) we derive from (5.17) the inequality

$$\begin{aligned}(\dot v(t), Pv(t))_{-1,1} + \lambda (v(t), Pv(t))_0 + F(v(t),\zeta(t)) +\ G\,(v(t),\zeta(t))&\\ -\,\delta\|Dv(t) + E\zeta(t)\|_Z^2 + p(t) \leq 0\,, \quad \text{a.a. } t > 0\,.&\end{aligned} \tag{5.18}$$

Integration of (5.18) on the time interval $0 < s < t$ gives

$$\begin{aligned}&\frac{1}{2}(v(t), Pv(t))_0 - \frac{1}{2}(v(s), Pv(s))_0 + \lambda \int_s^t (v(\tau), Pv(\tau))_0 d\tau + \int_s^t F(v(\tau),\zeta(\tau))d\tau\\ &+ \int_s^t G(v(\tau),\zeta(\tau))d\tau + \int_s^t p(\tau)d\tau \leq \delta \int_s^t \|Dv(t) + E\zeta(\tau)\|_Z^2 d\tau\,.\end{aligned} \tag{5.19}$$

From the inequalities (2.7) and (2.8) it follows that

$$\int_s^t F(v(\tau),\zeta(\tau))d\tau \geq 0 \tag{5.20}$$

and

$$\int_s^t G(v(\tau),\zeta(\tau))d\tau \geq \frac{1}{2}\Big[\Phi\big(Cv(t)\big) - \Phi\big(Cv(s)\big)\Big] + \lambda \int_s^t \Phi(Cv(\tau))d\tau\,, \quad 0 < s < t\,. \tag{5.21}$$

Taking into account now (5.19) – (5.21) we obtain that

$$\begin{aligned}&\frac{1}{2}(v(t), Pv(t))_0 + \frac{1}{2}\Phi(Cv(t)) - \frac{1}{2}(v(s), Pv(s))_0 - \frac{1}{2}\Phi(Cv(s))\\ &+2\lambda \int_s^t \Big[\frac{1}{2}\big(v(\tau), Pv(\tau)\big)_0 - \frac{1}{2}\Phi\big(Cv(\tau)\big)\Big]d\tau + \int_s^t p(\tau)d\tau \leq \delta \int_s^t \|Dv(\tau) + E\zeta(\tau)\|_Z^2\,.\end{aligned} \tag{5.22}$$

Now, we conclude that (5.22) implies the inequality (5.12) with the functions $m(\cdot), p(\cdot)$ and $g(\cdot)$ defined by (5.13) – (5.15) .

Remark 5.1 The frequency-domain condition **(A4)** depends on imbedding properties of the Sobolev spaces under consideration. Assume, for example, that $G \equiv 0, E = 0$ and

$$F(y,\xi) = \beta_0 \|y\|_0^2 - \beta_1 \|y\|_1^2\,, \quad (y,\xi) \in Y_0 \times \Xi\,, \tag{5.23}$$

where β_0 and β_1 are certain real constants.
In order to verify (5.10) we introduce the frequency-domain characteristic

$$\chi(i\omega) := (i\omega I^c - A^c_\lambda)^{-1} B^c \tag{5.24}$$

for $\omega \in \mathbb{R}$ s. t. $i\omega \in \rho(A^c_\lambda)$, where $A^c_\lambda := A^c + \lambda I^c$.
It follows that the frequency-domain condition (5.10) is satisfied if

$$\beta_0 \|\chi(i\omega)\xi\|^2_{Y_0^c} - \beta_1 \|\chi(i\omega)\xi\|^2_{Y_1^c} - \delta \|D^c \chi(i\omega)\xi\|^2_{Z^c} \le 0\,, \qquad \forall\, \xi \in \Xi^c\,, \quad \forall\, \omega \in \mathbb{R}: \quad i\omega \in \rho(A^c_\lambda)\,. \tag{5.25}$$

Suppose now that from the imbedding $Y_1^c \subset Y_0^c \subset Y_{-1}^c$ and the properties of the observation operator D we have the a priori estimate

$$\|v\|^2_{Y_0^c} \le c_1 \|v\|^2_{Y_1^c} + c_2 \varepsilon_{D^c} \|D^c v\|^2_{Z^c}\,, \quad \forall v \in Y_1^c\,, \tag{5.26}$$

where $c_1 > 0$ and $c_2 > 0$ are certain constants and

$$\varepsilon_{D^c} = \varepsilon_{D^c}(Y_1^c, Y_0^c) := \sup\,\{\|w\|_{Y_0^c} : w \in Y_1^c\,,\, D^c w = 0_{Z^c}\,, \|w\|_{Y_1^c} \le 1\} \tag{5.27}$$

is the *completeness defect* of the observation operator D^c with respect to the imbedding $Y_1^c \subset Y_0^c$.
It follows from (5.26) that the frequency-domain condition (5.25) is satisfied if

$$\beta_0\, c_1 \|\chi(i\omega)\xi\|^2_{Y_1^c} - \beta_1 \|\chi(i\omega)\xi\|^2_{Y_1^c} + \beta_0 c_2 \varepsilon_{D^c} \|D^c \chi(i\omega)\xi\|^2_{Z^c} - \delta \|D^c \chi(i\omega)\xi\|^2_{Z^c} \le 0 \qquad \forall\, \xi \in \Xi^c\,, \quad \forall \omega \in \mathbb{R}: \quad i\omega \in \rho(A^c_\lambda)\,. \tag{5.28}$$

For (5.28) it is sufficient that

$$\beta_0\, c_1 - \beta_1 \le 0 \quad \text{and} \quad \beta_0\, c_2\, \varepsilon_{D^c} - \delta \le 0\,. \tag{5.29}$$

We see that if $\beta_0 c_1 - \beta_1 \le 0$ the second condition of (5.29) is always satisfied if the completeness defect of the observation operator is small. In this case, assuming that the other assumptions for the Theorem 5.1 are also satisfied, it follows that the observation $z(t) = Dy(t)$ is determining for the output stability.
Suppose that $D_k y := (l_1(y), \ldots, l_k(y))$, where $l_i : Y_1 \to \mathbb{R},\ i = 1, \ldots, k$, are continuous linear functionals and $Y_1 = W^{s,2}(\Omega), Y_0 = W^{\sigma,2}(\Omega)$ with $s > \sigma$. Then $\varepsilon_{D^c} \approx c_1 (\frac{c_2}{k})^{s-\sigma}$, i.e., the completeness defect of the observation operator D_k depends on the smoothness properties of the imbedding $Y_1^c \subset Y_0^c$ (Triebel [14]). □

6 Determining observations for second-order visco-elastic contact problems

A typical frictional contact problem is modeled by the following second-order evolutionary variational inequality (Duvant, Lions [4], Han, Sofonea [8], Jarušek, Eck [9]): Find a displacement function u such that for a.a. $t \in [0, T]$

$$
\begin{aligned}
&(\ddot{u}(t), v - \dot{u}(t))_{\mathcal{V}_{-1},\mathcal{V}_1} + (\mathcal{A}\,\dot{u}(t), v - \dot{u}(t))_{\mathcal{V}_{-1},\mathcal{V}_1} \\
&+ \Big(g\big(u(t)\big), v - \dot{u}(t)\Big)_{\mathcal{V}_{-1},\mathcal{V}_1} + j(v) - j\big(\dot{u}(t)\big) \geq 0\,, \quad \forall\, v \in \mathcal{V}_1\,,
\end{aligned}
\tag{6.1}
$$

$$
u(0) = u_0 \in \mathcal{V}_1\,,\ \dot{u}(0) = v_0 \in \mathcal{V}_0\,. \tag{6.2}
$$

Here $\mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1}$ is a Hilbert space rigging structure, $\mathcal{A} : \mathcal{V}_1 \to \mathcal{V}_{-1}$ is a linear continuous operator which is called *viscosity operator*.
The nonlinear map $g : \mathcal{V}_1 \to \mathcal{V}_{-1}$ is the *elasticity operator* and $j : \mathcal{V}_1 \to \mathbb{R}_+$ represents the *contact functional.*
Under a solution u of (6.1), (6.2) on $(0, T)$ we understand a function $u(\cdot) \in L^2(0, T; \mathcal{V}_1)$ such that $\dot{u}(\cdot) \in L^2(0, T; \mathcal{V}_1), \ddot{u}(\cdot) \in L^2(0, T; \mathcal{V}_1\,, \int_0^T j(\dot{u}(\tau))d\tau < \infty$, and (6.1), (6.2) is satisfied for a.a. $t \in (0, T)$.
Let us assume that for any $(u_0, v_0) \in \mathcal{V}_1 \times \mathcal{V}_0$ and any time $T > 0$ a solution of (6.1), (6.2) exists. In order to rewrite (6.1), (6.2) as a first-order variational inequality (2.4), (2.5) we define the product Hilbert space rigging structure $Y_1 \subset Y_0 \subset Y_{-1}$ with

$$
Y_0 = \mathcal{V}_1 \times \mathcal{V}_0\,, \quad Y_1 = \mathcal{V}_1 \times \mathcal{V}_1\,, \quad Y_{-1} = \mathcal{V}_0 \times \mathcal{V}_{-1}\,. \tag{6.3}
$$

Let us introduce the new variables $y_1 = u\,, \ y_2 = \dot{u}$ and $\eta_2 = v$. It follows that $\dot{y}_1 = y_2$ and $\dot{y}_2 = \ddot{u}$. In this notation the variational inequality (6.1) can be rewritten as

$$
\begin{aligned}
&(\dot{y}_2, \eta_2 - y_2)_{\mathcal{V}_{-1},\mathcal{V}_1} + (\mathcal{A}y_2, \eta_2 - y_2)_{\mathcal{V}_{-1},\mathcal{V}_1} + \big(g(y_1), \eta_2 - y_2\big)_{\mathcal{V}_{-1},\mathcal{V}_1} \\
&\qquad + j(\eta_2) - j(y_2) \geq 0\,, \quad \forall\, \eta_2 \in \mathcal{V}_1\,,
\end{aligned}
\tag{6.4}
$$

Using the product topology we get for arbitrary $y = (y_1, y_2) \in Y_{-1} = \mathcal{V}_0 \times \mathcal{V}_{-1}$ and $\eta = (\eta_1, \eta_2) \in Y_1 = \mathcal{V}_1 \times \mathcal{V}_1$ the representation of the duality pairing on $Y_{-1} \times Y_1$ as

$$
(y, \eta)_{-1,1} = (y_1, \eta_1)_{\mathcal{V}_1} + (y_2, \eta_2)_{\mathcal{V}_{-1},\mathcal{V}_1}\,. \tag{6.5}
$$

It follows from (6.5) that

$$
(\dot{y}_2, \eta_2 - y_2)_{\mathcal{V}_{-1},\mathcal{V}_1} = (\dot{y}, \eta - y)_{-1,1} - (y_2, \eta_1 - y_1)_{\mathcal{V}_1}\,. \tag{6.6}
$$

A linear bounded operator $A : Y_1 \to Y_{-1}$ is defined by

$$
\begin{aligned}
&(-Ay, \eta - y)_{-1,1} = -(y_2, \eta_1 - y_1)_{\mathcal{V}_1} + (\mathcal{A}y_2, \eta_2 - y_2)_{\mathcal{V}_{-1},\mathcal{V}_1}\,, \\
&\qquad \forall\, y = (y_1, y_2), \eta = (\eta_1, \eta_2) \in Y_1 = \mathcal{V}_1 \times \mathcal{V}_1\,.
\end{aligned}
\tag{6.7}
$$

It is easy to see that A defined by (6.7) has the representation

$$A = \begin{bmatrix} 0 & I \\ 0 & -\mathcal{A} \end{bmatrix} . \tag{6.8}$$

In order to determine the linear operator $B : \Xi = \mathcal{V}_1 \to Y_{-1}$ we use the equation

$$\begin{aligned} (-B\varphi(y_1), \eta - y)_{-1,1} &= (\varphi(y_1), \eta_2 - y_2)_{\mathcal{V}_{-1},\mathcal{V}_1}, \\ &\forall\, y = (y_1, y_2), \eta = (\eta_1, \eta_2) \in Y_1 = \mathcal{V}_1 \times \mathcal{V}_1 . \end{aligned} \tag{6.9}$$

From (6.9) it follows that

$$B\varphi(Cy) = \begin{bmatrix} 0 \\ -\varphi(y_1) \end{bmatrix} , \tag{6.10}$$

where the linear operator $C : Y_1 \to W := \mathcal{V}_1$ is defined by $(y_1, y_2) \mapsto y_1$.

The last remainig element in the inequality (2.4) is the contact functional $\psi : Y_1 \to \mathbb{R}_+$ given by

$$\psi(y) := j(y_2) , \quad \forall\, (y_1, y_2) \in Y_1 = \mathcal{V}_1 \times \mathcal{V}_1 . \tag{6.11}$$

References

[1] Brusin, V. A., *The Luré equations in Hilbert space and its solvability.* Prikl. Math. Mekh. **40** 5, 947 – 955 (1976) (in Russian).

[2] Chueshov, I. D., Introduction to the Theory of Infinite-Dimensional Dissipative Systems. ACTA. Kharkov, (1999).

[3] Chueshov, I. D., *On functionals that uniquely determine the long-time behaviour of solutions to von Karman evolution equations.* Uspekhi Matem. Nauk **51** 5, 162 (1996).

[4] Duvant, G. and J.-L. Lions, Inequalities in Mechanics and Physics. Springer-Verlag, Berlin (1976).

[5] Foias, C. and G. Prodi, *Sur le comportement global des solutions nonstationaires des équations de Navier-Stokes en dimension deux.* Rend. Sem. Mat. Univ. Padova **36**, 1 – 34 (1967).

[6] Foias, C. and R. Temam, *Determination of solutions of the Navier-Stokes equations by a set of nodal values.* Math. Comput. **43**, 117 – 133 (1984).

[7] Hoffmann, K.-H. and J. Sprekels, *On the identification of parameters in general variational inequalities by asymptotic regularization.* SIAM J. Math. Anal., **17**, 1198 – 1217 (1986).

[8] Han, W. and M. Sofonea, *Evolutionary variational inequalities arising in viscoelastic contact problems.* SIAM J. Numer. Anal. **38** 2, 556 – 579 (2000).

[9] Jarušek, J. and Ch. Eck, *Dynamic contact problem with friction in linear viscoelasticity.* C. R. Acad. Sci. Paris **322**, 497 – 502 (1996).

[10] Ladyzhenskaya, O. A.,*A dynamical system generated by the Navier-Stokes equations.* J. Soviet Math. **3** 4, 458 – 479 (1975).

[11] Likhtarnikov, A. L. and V. A. Yakubovich, *The frequency theorem for equations of evolutionary type.* Siberian Math. J. **17** 5, 790 – 803 (1976).

[12] Lions, J.-L., Quelques Méthodes de Résolution des Problémes aux Limites nou Linéaires. Dunad, Paris (1969).

[13] Maksimov, V. I., *Inverse problems for variational inequalities.* Intern. Series of Numerical Math., Birkhäuser Verlag, Basel, **107**, 275 – 286 (1992).

[14] Triebel, H., Interpolation Theorie, Function Spaces, Differential Operators. Amsterdam, North-Holland (1978).