The use of Boltzmann's transport equations and Lax-Phillips scattering semigroups in time-series analysis

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1. Introduction

A time-series is considered as element of a function space (weighted L^p space, Sobolev space). We assume that this time-series is *causal*, i.e. it is the output (outgoing wave) of a nonlinear nonautonomous dynamical system depending on a certain input (control or incoming wave). Under very general conditions such an input / output control system for concrete mechanical problems can be described by an unknown nonlinear Volterra integral equation ([1, 18, 14, 5]). For this integral equation we derive a universal abstract nonautonomous dynamical system in the spirit of Kalman's ([8]) and Salamon's ([16]) representation theory with physically motivated rigged Hilbert spaces as phase spaces. As canonical representation we can take, according to this theory, PDE's ([4]) or equations with delay ([3]).

The universal abstract system can be considered as generalized nonlinear transport equation of Boltzmann type ([15]). The system consists of a linear part (collisionless transport operator) and a nonlinear part (scattering operator). Any such controllable and observable linear part generates a continuous semigroup which can be transformed into a Lax-Phillips model. Such a transformation is very useful for inverse problems, i.e. the determination of parameters from time-series. Note, however, that in difference to other inverse problem techniques ([1, 18, 14, 5]) in our approach not the parameters of the generating equation, but some parameters of the abstract transport equation are estimated. According to the abstract absolute stability theory ([20]) many global stability problems are determined by whole classes of nonlinear systems whose nonlinearities are characterized by certain quadratic forms. The aim of our method is to receive from time-series some information of this class. The number of parameters for the description of such a class is in general much smaller than the number of unknown parameters of the concrete equation. A new method for recurrent learning of input-output behaviour based on absolute stability criteria, is considered in [17]. In difference to our approach the authors of [17] introduce network equations from absolute stability theory containing some weights which are estimated by minimizing some error functional. The main property of the adapted network is its internal stability.

The description of the linear part is done, as usual in the control theory, by the frequency-domain characteristic. The nonlinear part is given by a quadratic form over the rigged Hilbert spaces and the control space. Using a frequency-domain theorem of Brusin ([4]) for the solvability of a *linear* Riccati-operator equation, a cone in the space of perturbations is constructed, which contains the unstable perturbations. As an example we consider a scalar ODE of the second order. An associated PDE transport equation of Bolzmann-type will be derived which has the same global stability properties as the ODE problem.

2. Realization of a Volterra integral equation as control system

Consider the nonlinear Volterra integral equation

$$\sigma(t) = h(t) + \int_0^t G(t-\tau)\varphi(\sigma(\tau),\tau)\,\mathrm{d}\tau \tag{1}$$

with $\sigma : \mathbb{R} \to \Xi (= \mathbb{R}^n)$ as time-series or output ,

$h : \mathbb{R} \to \Xi$ as perturbation ,

$u(\cdot) := \varphi(\sigma(\cdot),\cdot) : \mathbb{R} \to U \ (= \mathbb{R}^n)$ as control ,

$G(t) \in \mathrm{Lin}(U,\Xi)$ as kernel and

$\varphi : \Xi \times \mathbb{R} \to U$ as nonlinearity .

Assumption 1.

Ξ, U are real Hilbert spaces ,

$\exists P = P^* \in \mathrm{Lin}(\Xi,\Xi),\ Q \in \mathrm{Lin}(U,\Xi),\ R \in \mathrm{Lin}(U,U)$:

$$(\sigma(t), P\sigma(t))_\Xi + 2(\sigma(t), Q\varphi(\sigma(t),t))_\Xi + (\varphi(\sigma(t),t), R\varphi(\sigma(t),t))_U \le 0 \tag{2}$$

$\forall \sigma(\cdot), \varphi(\sigma((\cdot),\cdot)\,, \sigma(\cdot)$ continuous solution from (1)

(Quadratic constraints)

Assumption 2.

The linear part of (1) is ρ-stable, i.e.

$\exists \rho \ge 0\ \forall u \in L^2_\rho \mapsto \sigma(\cdot) \in W^{1,2}_\rho$ is a bounded operator and

$L^2_\rho(\mathbb{R}_+;\Xi) := \{f \in L^2_{loc} : \int_0^\infty |f(t)|^2_\Xi e^{2\rho t}\,\mathrm{dt} < \infty\}$ is a weighted $\mathrm{L^p}$-space ,

$W^{1,2}_\rho(\mathbb{R}_+;\Xi) := \left\{ f \in L^2_\rho(\mathbb{R}_+;\Xi); \dot{f} \in L^2_\rho(\mathbb{R}_+;\Xi) \right\}$ is a Sobolev space ,

$\sigma(t) = \int_0^t G(t-\tau)u(\tau)\,\mathrm{d}\tau$.

Goal: Find Hilbert spaces $Z_1 \subset Z_0 \subset Z_{-1}$ (Rigged Hilbert space structure)
Hilbert spaces U, Ξ and linear bounded operators
$A : Z_1 \to Z_{-1},\ B : U \to Z_{-1},\ C : Z_0 \to U$
such that the qualitative behaviour of (1) coincides with the qualitative behaviour of the nonautonomous dynamical system

$$\begin{aligned} \dot{z} &= Az + Bu(t) \\ \sigma = Cz,\ u(t) &= \varphi(\sigma(t), t)\,, \end{aligned} \tag{3}$$

where the solution $z(\cdot, z_0, u)$ with $z(0, z_0, 0) = z_0$ satisfies the time-invariance or cocycle property [2]

$z(t+s, z_0, u) = z(t; z(s; z_0, u), \tau^s u)$

$\sigma(t+s, z_0, u) = \sigma(t; z(s; z_0, u), \tau^s u)\,, \quad \forall t, s \geq 0\,,$

with $\tau^s u(t) := u(t+s)$ as shift operator.

We call this

imbedding of the time-series or of (1) into a time-invariant control system.

Theorem 1.

(Realization theorem of Kalman ([8]), Helton ([7]), Salamon ([16]))

Suppose the linear part of the input / output process given by (1) is ρ-stable. Then there exists an imbedding of (1) into a system (3) by the transport equation, i.e. by a system (3)

$$\text{with } Z_0 := W^{1,2}_\rho,$$

$$Z_1 := D(A) = \{\xi : \xi(s) \in W^{1,2}_\rho, \int_0^\infty e^{2\rho s} |\ddot{\xi}(s)|^2 \mathrm{d}s < \infty\},$$

$$(A\xi)(s) := \frac{\partial \xi(s)}{\partial s} \text{ transport operator}$$

$$(B\eta)(s) := G(s)\eta,\ \eta \in U = (\mathbb{R}^n),\ Cz(s) := z(0).$$

Example 1

The nonlinear Boltzmann transport equation from scattering theory

$$\frac{\partial \sigma}{\partial t} = \frac{\partial \sigma}{\partial x} + \int\limits_{-\infty}^{x} G(x-s)\varphi(\sigma(\tau,x))\,\mathrm{d}\tau\,.$$

Suppose that we know G, P, Q and R. Than the abstract evolution system gives the following information

Theorem 2.

(Generalized Brusin's theorem) ([4])

Let $\hat{G}(\lambda) := \int\limits_{0}^{\infty} e^{-\lambda t} G(t)\,\mathrm{dt}$ be the transfer operator. Suppose that the frequency–domain condition

$$\hat{G}^*(i\omega)PG(i\omega) + 2Re(Q^*G(i\omega)) + R > 0 \quad \forall \omega \in \mathbb{R} \tag{4}$$

is satisfied.

1) Then there exists an operator $M = M^* : W^{1,2}_\rho \rightarrow W^{1,2}_\rho$ with the following properties:

Suppose $(\sigma(\cdot), h(\cdot))$ satisfies (1) and $(h(\cdot), Mh(\cdot)) < 0$. Then $\sigma(\cdot) \in L^2(\mathbb{R}_+;\Xi)$, i.e. it is stable. If $(h(\cdot), Mh(\cdot)) > 0$ then $\sigma(\cdot)$ is unstable, i.e. there exists a number $\beta > 0$ such that

$$\lim_{T\to\infty} e^{-\beta T} \int_0^T \mid \varphi(\sigma(t), t) \mid^2 \mathrm{dt} = \infty\,.$$

2) M is the operator solution of a linear integral equation.

3. Realization as Lax-Phillips scattering model

Let U, Ξ, Z be Hilbert spaces and

$$H := \underbrace{L^2(-\infty, 0; U)}_{\text{outgoing space}} \oplus \underbrace{Z}_{\text{inner state space}} \oplus \underbrace{L^2(0, +\infty; \Xi)}_{\text{incoming space}}$$

Let $\mathbb{G}^t$ be a continuous and bounded semigroup of operators on H which describes how information in H changes in time.

Let τ_-^t (resp. τ_+^t) denote the translation by t to the right on $L^2(-\infty, 0; U)$ (resp. $L^2(0, \infty; U)$).

Denote by $S_{[-a,b]}$ the subspace

$L^2(-a, 0; U) \oplus Z \oplus L^2(0, b; \Xi)$

Let $\pi_{[-a,b]}$ be the orthogonal projection onto $S_{[-a,b]}$.

The semigroup $\mathbb{G}^t$ is of Lax-Phillips type if the following conditions are satisfied ([7]):

(i) $\quad \mathbb{G}^t \xi = \tau_+^t \xi, \ \forall \xi \in L^2(0, \infty; \Xi) \quad (\mathbb{G}^t)^* \xi = (\tau_-^t)^* \xi, \ \forall \xi \in L^2(-\infty, 0; U)$

(ii) $\quad$ For $a > 0$ define $\mathbb{W}^t := \pi_{[-\infty,a]} \mathbb{G}^t \pi_{[-a,\infty]}$

Then $\mathbb{W}^t$ and $(\mathbb{W}^t)^*$ are asymptotically stable.

(iii) $\quad$ For $a \geq 0$ we have

$$\lim_{t\to\infty} \pi_{[-a,\infty)} (\mathbb{G}^t)^* f = 0 \quad \lim_{t\to\infty} \pi_{[-a,\infty)} \mathbb{G}^t f = 0$$

Example 2.

Given a rod of unit length and temperature distribution

$z(x,t), \ z(0,t) = z(1,t) = 0$

$Z := L^2(0,1), \quad U \equiv \Xi := \mathbb{R}$

$Af := \dfrac{d^2 f}{dx^2}$ on $D(A)$ where

$$D(A) := \{f \in L^2(0,1) : \ f(0) = f(1) = 0, \ \int_0^1 \left| \frac{d^2 f}{dx^2} \right|^2 < \infty\}$$

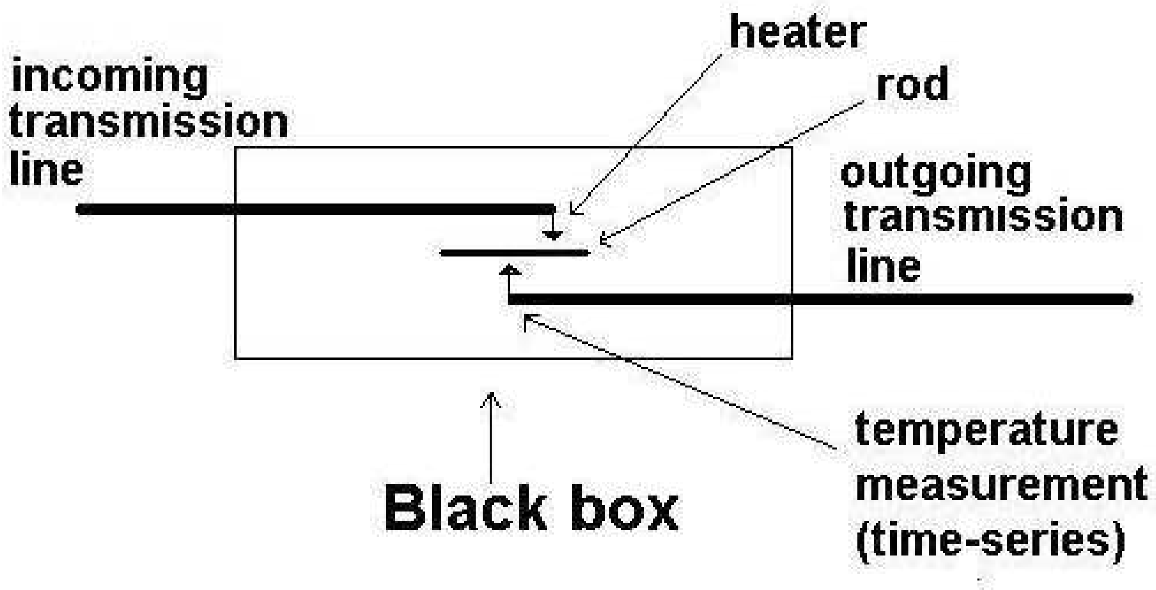

Figure 1: Temperature measurement of a heated rod as Lax Phillips scattering model (after [7])

There is a heater in the rod which after input α supplies heat with distribution $\alpha h(x)$ and a measuring instrument with reads the temperature at $x = \frac{1}{2}$ (see Fig. 1).

$\Rightarrow B\alpha := \alpha h(x),\ Cz := z(\frac{1}{2}, t)$ defines a system

$$\begin{aligned} \dot{z} &= Az + Bu \\ \sigma &= Cz\,. \end{aligned} \tag{5}$$

Advantage of a Lax-Phillips scattering model

Assume that (5) generates a general Lax-Phillips model. Then the input and output information in (5) completely determines the eigenvalues of A.

Theorem 3. (Lax-Phillips, [11])

Suppose that in (5) the pair (A, B) is controllable and the pair (A, C) is observable. Suppose also that the frequency-domain characteristic

$$\mathcal{X}(i\omega) = C(i\omega I - A)^{-1}B$$

can be extended to an meromorphic function. Then the poles of $\mathcal{X}(\cdot)$ are exactly the eigenvalues of A.

$\Rightarrow$

The inverse Lax-Phillips scattering problem (determination of parameters from observation) has a unique solution.

4. Transport equation for the Mathieu-Hill equation

We consider an ODE of the second order

$$\ddot{\sigma} + \alpha\dot{\sigma} + \varphi(\sigma(t), t) = 0 \tag{6}$$

with a smooth nonlinearity $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Assume that any solution of (6) exists on $\mathbb{R}$.

Let us rewrite (6) in the following way

$$\begin{cases} \dot{z}(t) = Az(t) + B\varphi(\sigma(t), t) \\ \sigma(t) = Cz(t)\,, \end{cases} \tag{7}$$

$$\text{with} \quad A = \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where $\sigma(t)$ is the input and $\varphi(\sigma(t), t)$ is the output.

As "nonlinear part" is considered the function

$$\varphi(\sigma, t) = (\beta + \gamma\cos(t))\sigma\,, \tag{8}$$

where β and γ are parameters. Note that equation (6) with φ given by (8) has the form of the Mathieu-Hill equation. It is well-known ([9, 13, 10]) that this equation with parametric exicitation can be used to describe some bifurcations in dynamical buckling processes.

Time is considered on the finite interval $[0, T]$.

All functions are considered as sequences

$$\{\sigma(t_i)\}_1^{N+1}, \; t_k = (k-1)\frac{T}{N}, k = 1, 2.., N+1$$

where $N + 1$ is the number of nodes on the interval $[0, T]$.

Step 1

Find a sector for the nonlinear part such that

$\mu_1 \leq \varphi(\sigma, t)/\sigma \leq \mu_2 \quad \forall t, \sigma.$

Take initial data $(\sigma_i(0), \dot{\sigma}_i(0)),\ i = 1, 2.., L,$ and calculate the numbers μ_1, μ_2 such that the relation

$\mu_1 \leq \varphi(\sigma(t_i), t_i)/\sigma(t_i) \leq \mu_2\,, \quad i = 1, .., N + 1\,,$

is satisfied. For the calculation of μ_1, μ_2 an adaptive algorithm is used which is finitely converging in the sense of Yakubovich ([6]).

Step 2

Write system (6) as Volterra integral equation

$$\begin{aligned} \sigma(t) &= h(t) + \int_0^t G(t-\tau)\varphi(\sigma(\tau),\tau)\,\mathrm{d}\tau\,, \\ \varphi(\sigma,t) &= (\beta+\gamma\cos(t))\sigma, \end{aligned} \tag{9}$$

where $h(\cdot)$ is the input and $\sigma(\cdot)$ the output ($\sigma \equiv \sigma_h$).

The goal is to construct an operator M which gives all information about stability of $\sigma_h(\cdot)$ with respect to the input $h(\cdot)$.

Assume that the kernel of (9) can be written as

$$G(t-\tau) = e^{\lambda(t-\tau)},$$

where λ is an unknown parameter.

Let $\rho \geq 0$ be the unknown parameter of the Hilbert space L^2_ρ introduced in Section 2.

Step 3

In order to construct the operator M we have as an auxiliary problem to solve the linear Fredholm integral equation of the second kind

$$\int_0^T S_{(\rho,\lambda)}(t,\tau)\tilde{u}_{h,(\rho,\lambda)}(\tau)\mathrm{d}\tau + \tilde{u}_{h,(\rho,\lambda)}(t) = g_{h,(\rho,\lambda)}(t), \tag{10}$$

where $S_{(\rho,\lambda)}$ is a function depending on ρ and λ, and $g_{h,(\rho,\lambda)}$ depends also on $h(\cdot)$.

From this equation we get $\tilde{u}_{h,(\rho,\lambda)}(\cdot)$ which will be used further.

Remark 1

If we solve the integral equation (10) we get the solution of an associated Riccati equation. In general the Riccati equation is a quadratic equation with respect to the unknown matrix or operator. In our situation this equation (10) is linear what is important for practical realization. The reason for this is the special type of hyperbolic equations arising in (3). This property was also investigated in [19].

Step 4

Construct the cost-functional $J^T_{\lambda,\rho}(\cdot)$ on L^2_ρ.

Take some initial values $\overline{\lambda}, \overline{\rho}$, calculate the functional with these parameters and compare with the data.

Use for this an optimization procedure with respect to λ, ρ for the functional computed along the solution of the Fredholm integral equation (10).

As result of this step we get the functional $J^T_{\lambda_0,\rho_0}$.

Step 5

Define the operator M^T by

$$(M^T h)(s) := -\frac{1}{\lambda_1}\int\limits_0^T \{e^{-2\rho(\tau)}\left[e_{s-\tau}e^{\lambda_1(s-\tau)} + e_{\tau-s}\mu_1(s-\tau)\right] + \\ +\mu_2(-\tau)e^{\lambda_1 s}\}(P\widetilde{\sigma}_h(\tau) + Qh(\tau))\,\mathrm{d}\tau,\ \forall h \in W^{1,2}_{\rho_0}, \qquad (11)$$

where $\widetilde{\sigma}_h(t) = \int\limits_0^t (e^{\lambda_0(t-\tau)} + h(\tau))\mathrm{d}\tau + h(t)$ and the functions $\lambda_1(\cdot), \mu_1(\cdot), \mu_2(\cdot)$ depend only on ρ_0.

Then the sign of the test functional

$$< M^T h, h >= \int\limits_0^T (M^T h)(s)h(s)e^{2\rho_0 s}\,\mathrm{ds} \qquad (12)$$

gives us the information about stability of $\sigma(\cdot)$ according to Brusin's theorem (Theorem 2).

5. Numerical results

Consider the equation (6),(7) with the system parameters

$$\alpha = 1/3;\ \beta = 1;\ \gamma = 2. \tag{13}$$

Using the above algorithm with $T = 2\pi, N = 18, L = 50$ we find the sector from Step 1 for the "nonlinearity" (8) with $\mu_1 = -1, \mu_2 = 3$.
For the kernel $G(\cdot)$ and the function space L^2_ρ we obtain the parameters

$$\lambda_0 = 0.29,\ \rho_0 = 0.1\ . \tag{14}$$

This defines the operator M^T for the test functional (11)
In order to verify our result we consider the solution of (9) with the initial data

$$\sigma(0) = 0.15683, \dot{\sigma}(0) = 0,25269\ . \tag{15}$$

Computing the associated h in (9) we get a positive sign of the test functional (11). According to Brusin's theorem the solution must be unstable. The direct calculation of the solution (Fig. 3) shows their instability.
This means that the information from test functional (12) is correct.

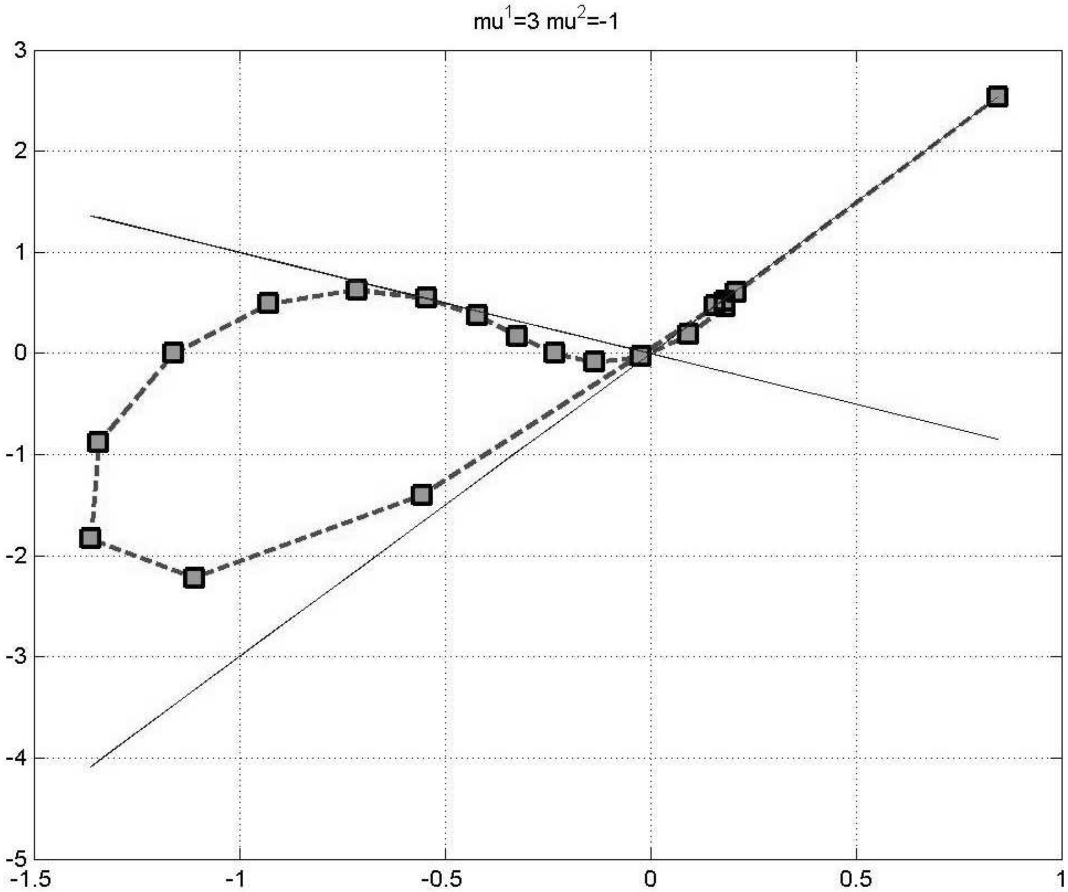

Figure 2: The sector for the nonlinearity (8)

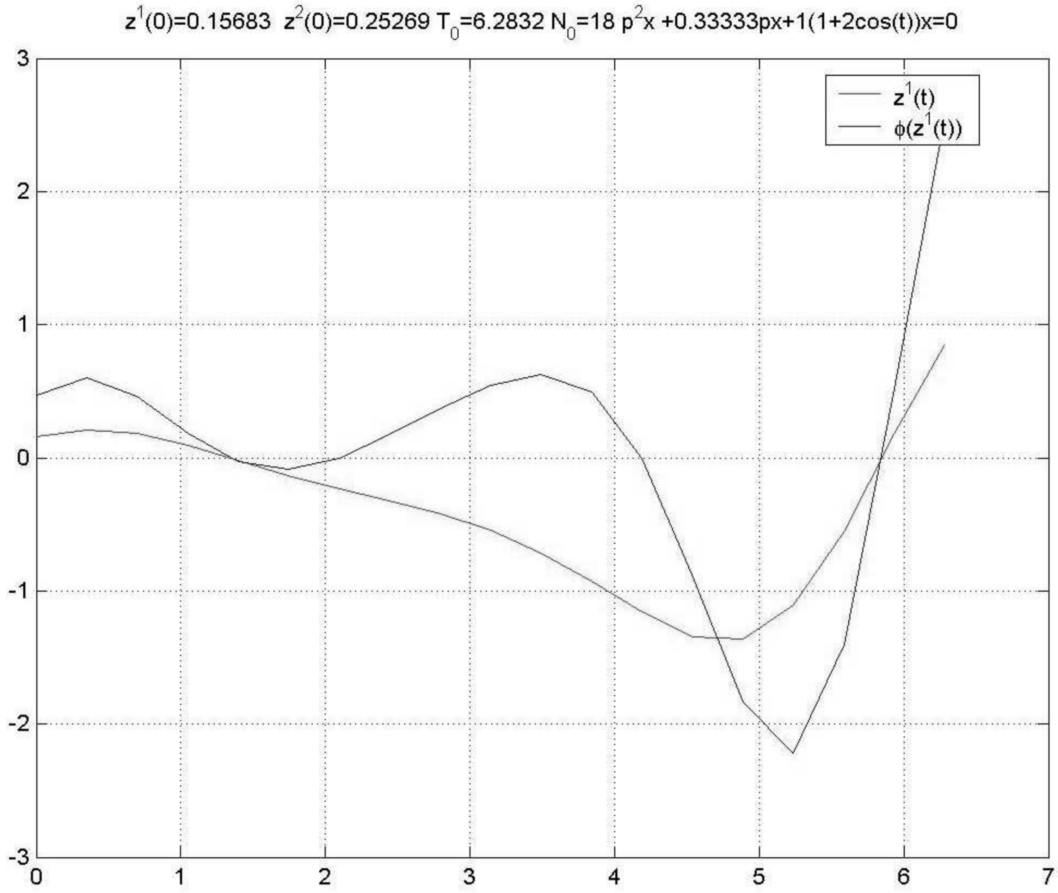

Figure 3: Solution component of (9) with initial condition (14)

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