Embedding techniques for nonholonomic constrained and forced systems with an application to the rolling elastic tire

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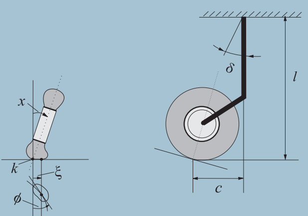
1. Nonholonomic constraint systems

1.1 The mechanical model



1.2 Class of constrained mechanical systems

Suppose the n-dimensional C^r-manifold Q is the *configuration manifold* of a mechanical system. A smooth curve $t \mapsto q(t) \in Q$ satisfies *linear constraints* if there is a *distribution* $\mathcal{D}$ on Q, i.e. a subbundle of the tangent bundle TQ such that $\dot q(t) \in \mathcal{D}(q(t))$ for all $t \in \mathbb{R}$. If this distribution is nonintegrable, the constraints are called *nonholonomic*. Consider the smooth *Lagrangian* $L : TQ \to \mathbb{R}$. Then the *constrained variational principle* is given by

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i}\right)\delta q^i = 0 \qquad (1.1)$$

(we use Einstein's summation convention) for all variations δq such that $\delta q \in \mathcal{D}(q)$ at each point q of the curve $t \mapsto q(t)$.
Let $\{\omega^a\}$ be a set of k independent 1-forms whose vanishing describes the nonholonomic constraints. Choose a chart and a basis for the constraints such that

$$\omega^a(q) = ds^a + A^a_\alpha(r,s)\,dr^\alpha, \quad a = 1, 2, \ldots, k, \qquad (1.2)$$

where $q = (r, s) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. We define the constrained *Lagrangian* L_c by substituting the constraints $\dot s^a = -A^a_\alpha \dot r^\alpha$, $a = 1, 2, \ldots, k$, into the Lagrangian:

$$L_c(r^\alpha, s^a, \dot r^\alpha) := L(r^\alpha, s^a, \dot r^\alpha, -A^a_\alpha(r,s)\dot r^\alpha).$$

Then the equation of motion for the mechanical system with linear constraints can be written as

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot r^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot s^b} B^b_{\alpha\beta}\dot r^\beta \qquad (1.3)$$

where

$$B^b_{\alpha\beta} := \frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{\partial A^b_\alpha}{\partial s^a}.$$

1.3 The pneumatic tire as nonholonomic system

The equation of motion is given by the Lagrange d'Alembert equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q^j} - \frac{\partial T}{\partial q^j} = Q_j + R_j,\ j = 1, \ldots, n, \qquad (1.4)$$

where $T = T(q^j, \dot q^j, t)$ is the kinetic energy, $Q_j = Q_j(q^i, \dot q^i, t)$, $j = 1, 2, \ldots, n$, are the generalized forces, $R_j = R_j(\xi, \varphi, \chi)$ are the generalized reaction forces of the constraints *connected with the elastic deformation* of the tire. The linear constraints are given by the rolling property of the tire

$$\dot x + \dot\xi + v\theta + v\varphi = 0, \quad \dot\theta + \dot\varphi - \alpha v\xi + \beta v\varphi + \gamma v\chi = 0, \qquad (1.5)$$

where α, β and γ are constants. These constraints are nonholonomic because the 1-forms $\omega^1 := dx + d\xi + v(\theta + \varphi)dt$, $\omega^2 := d\theta + d\varphi + v(-\alpha\xi + \beta\varphi + \gamma x)dt$ are not integrable.
Substitution of (1.5) into (1.4) gives a general nonautonomous system

$$\dot q = f(q, u(t)) \qquad (1.6)$$

with $q \in Q$, $u(t) \in U \subset \mathbb{R}^{d_u}$ as perturbation or control. As a typical property of (1.6) the set of *equilibria* for $u(t) \equiv 0$ is a continuum and all they are unstable. Possibility of unstable oscillations.

Example 1.1
Bicycle (**Neimark, Fufaev** (1972); **Getz, Marsden** (1995))

$$\ddot\chi - \nu\chi - A\dot\psi - B\psi = 0,$$
$$\ddot\psi + \tau\dot\psi + \dot\chi = 0, \qquad \nu,\ A,\ B,\ \tau - \text{parameters},$$

χ - angle between bicycle plane and the vertical,
ψ - angle of rotation of the steering wheel.

Stationary set: $E = \{(\chi, \psi) \in \mathbb{R}^2 \mid \nu\chi + B\psi = 0\}$, E is a continuum with $\dim E = 1$.

Routh-Hurwitz: E is stable if $\tau > \frac{B}{A}$ and $\tau\nu < B$.

Symmetries: The rigid frame of the bicycle is assumed to be symmetric about the plane containing the rear wheel.

Controlled bicycle: $\ddot\chi - \nu\chi - A\dot u - Bu = 0$
control $u = \alpha\chi + \beta\dot\chi + \gamma\int_0^t \chi\, d\tau$,
α, β, γ parameters.

2. Embedding techniques

2.1 Strong differential observability

Given

$$\dot q = f(q, u),\ w = h(q) \qquad (2.1)$$

on the n-dimensional analytical compact manifold Q, $u : \mathbb{R} \to U \subset \mathbb{R}^{d_u}$ is a smooth control, $h : Q \to \mathbb{R}^{d_w}$ a smooth output. Let $k \in \mathbb{N}$. Define the map $\phi_{f,h,k} : Q \times U \times \mathbb{R}^{(k-1)d_u} \to \mathbb{R}^{kd_w}$ by

$$(q_0, u(0), \dot u(0), \ldots, u^{(k-1)}(0)) \mapsto \qquad (2.2)$$
$$(h(\varphi(t)), \frac{d}{dt}h(\varphi(t)), \ldots, \frac{d^{k-1}}{dt^{k-1}}h(\varphi(t))|_{t=0}$$

where $\varphi(\cdot)$ is the solution of (2.1) with $\varphi(0) = q_0$. Consider the *suspension* of $\phi_{f,h,k}$ as the map

$$S\phi_{f,h,k} : Q \times U \times \mathbb{R}^{(k-1)d_u} \to \mathbb{R}^{kd_w} \times \mathbb{R}^{kd_u}$$

given by $(q_0, u(0), \dot u(0), \ldots, u^{(k-1)}(0)) \mapsto (h(\varphi(t)), \frac{d}{dt}h(\varphi(t)), \ldots, \frac{d^{k-1}}{dt^{k-1}}h(\varphi(t))|_{t=0},$
$u(0), \dot u(0), \ldots, u^{(k-1)}(0))$.

System (2.1) is called *strongly differentially observable of order* k if $S\phi_{f,h,k}$ is an embedding.

Gauthier, Kupka (1996): $S\phi_{f,h,k}$ is for $k \geq 2\dim Q + 1$ generically (in a precise sense) an embedding if $d_w > d_u$ and Q is a compact analytic manifold.

2.2 Equilibrium points and observability

Joan (1995): Let Q, f, h be analytic, $u(t) \equiv 0$, (2.1) strongly differentially observable and E be the set of equilibria. Then $\dim E \leq d_w - 1$.

Conjecture 2.1 Let in (2.1) f be an analytic vector field on the compact analytic manifold Q with the stationary set E satisfying

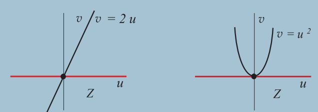
$$d_w > \max\{\dim E + 1, d_u\}. \qquad (2.2)$$

Then the set of analytic functions $h : Q \to \mathbb{R}^{d_w}$ such that (2.1) is strongly differentially observable of order $k \geq 2\dim Q + 1$ contains a residual set of the analytical functions $h : Q \to \mathbb{R}^{d_w}$.

2.3 Observability and transversality

The observability property is expressed in terms of the *transversality* of a particular mapping. Let M, N be smooth manifolds, Z be a submanifold in N and $g : M \to N$ be a smooth map. The map g *transversally intersects* Z at $p \in M$ if either $q = g(p) \notin Z$ or, if $q \in Z$, then
Image $(d_p g) + T_q(Z) = T_q(N)$.

Notation: $g \bar{\pitchfork}_p Z$



$g \bar{\pitchfork}_0 Z$ $\qquad$ g does not intersect Z transversally at 0

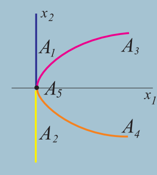
Transversality is a generic property (open and dense).

Analytic (subanalytic) sets are locally defined by a finite number of equations (equations and inequalities) given by analytic functions.

Whitney stratification: Decomposition of a set A into a finite union of manifolds A_i given by algebraic equations or inequalities.

Example 2.1

$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1(x_1 - x_2^2) = 0\} = \bigcup_{i=1}^5 A_i$,
where A_1 (A_2) is the positive (negative) part of the x_2-axis,
$A_3 = \{(x_1, x_2) | x_2 = x_1^{1/2}, x_1 > 0\}$,
$A_4 = \{(x_1, x_2) | x_2 = -(x_1)^{1/2}, x_1 > 0\}$,
$A_5 = \{(0, 0)\}$.

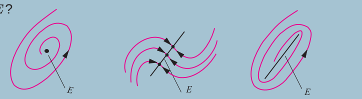


Bad sets (no transversality) are vector bundles with analytic (subanalytic) subsets of a vector space as a typical fibre.

3. Output stabilization

3.1 The inverse Lyapunov theorem

Given (2.1) with the stationary set E. Can we find a control u which *depends only on the measurement* w and *which stabilizes* E?



Convergence to a single equilibrium $\qquad$ Pointwise convergence to the stationary set $\qquad$ Convergence to the stationary set

Gauthier, Kupka (1994)

Assumption (A1): For $u(t) = u_0(t)$ the set E is asymptotically stable.

Inverse Lyapunov's theorem (**Wilson** (1969)): Suppose $D = \{q \in Q | \varphi^t(q) \to E\}$ is the domain of attraction. Then there exists a C^∞ function $V : D \to \mathbb{R}$ such that

1) $V(q) = 0, \forall q \in E$, $V(q) > 0, \forall q \in D \setminus E$;
2) $\dot V_f(q) < 0$ in $D \setminus E$; **3)** $V(q) \to \infty$ as $q \to \partial D$.

Goal: Use such a Lyapunov function in order to construct a stabilizing feedback for the autonomous system (2.1)

3.2 Center manifold theorem

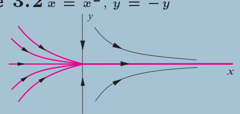
Let M be an open set in $\mathbb{R}^n$, f a C^∞ vector field on M, and $p \in M$ a stationary point of f. Denote by $\{\varphi^t(\cdot)\}_{t \in (-\varepsilon, \varepsilon)}$ the local flow of f on $(-\varepsilon, \varepsilon) \times U$, U a neighborhood of p.
Let $\phi^t : T_p\mathbb{R}^n \to T_p\mathbb{R}^n$, $t \in (-\varepsilon, \varepsilon)$, be the tangent mapping of $\{\varphi^t\}$ at p.

Assumption (A3): $\mathbb{R}^n = N \oplus H$ is a df-invariant decomposition, such that $df_{|N}$ has only imaginary eigenvalues, and $df_{|H}$ has no purely imaginary eigenvalues.

Center manifold theorem (**Carr** (1981))
For any $r \in \mathbb{N} \cup \{\infty\}$ there exists an open neighborhood U of p, a C^r closed submanifold Z of U such that:

(i) $p \in Z$ and $T_pZ = N$, and **(ii)** for any $p \in Z$, the maximal orbit of f in U passing through p at time 0 is contained in Z.
(iii) For any $q \in U$ such that the maximal positive (resp. negative) semiorbit of f in U starting (resp. ending) for $t = 0$ at q, is defined for all $t \geq 0$ (resp. $t \leq 0$), then the set $\omega_U(q)$ (resp. $\alpha_U(q)$) is contained in Z.

Example 3.2 $\dot x = x^2$, $\dot y = -y$



Center manifold

3.3 Generically asymptotic observer

Assumption (A2): System (2.1) is strongly differentially observable of order k. Define the *high-gain matrix*
$K_\Theta := \mathrm{diag}(\Theta, \Theta^2, \ldots, \Theta^n)$, $\Theta > 1$ parameter,

$$A_{k,m} := \begin{pmatrix} 0I_m & 00 \\ \ldots & \ldots \\ 0 & 0I_m \\ 0 & 00 \end{pmatrix}, \quad b_{k,m} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ I_m \end{pmatrix},$$

$C_{k,m} := (I_m, 0, \ldots, 0)$, the *stabilizing feedback* $\alpha_k(\cdot,\cdot)$ of the k-th extension of (2.1), the *phase-variable representation* $\phi_k(\cdot,\cdot,\cdot)$ (both available), the *k-th extended system*

$$\left.\begin{array}{l} \dot q = f(q, u^{(0)}),\ \omega = (u^{(0)}, \tilde u_{k-1}), \\ \dot\omega = A_{k,d_u}\omega + b_{k,d_u}\alpha_k(z, \omega), \end{array}\right\} \qquad (3.1)$$

the *output observer in the Luenberger form for state estimation*

$$\dot z = (A_{k,k_w} - K_\Theta C_{k,d_w})z \qquad (3.2)$$
$$+ K_\Theta h(q) + b_{k,d_w}\phi_k(z, \omega, \alpha_k(z, \omega)).$$

Conjecture 3.1 Suppose that the assumptions **(A1)**, **(A2)** are satisfied. Then system (3.1), (3.2) gives an asymptotic stabilization of the stationary set E, i.e. dist $(q(t), E) \to 0$ as $t \to \infty$.

Corollary 3.1 If $k > 2\dim Q + 1$ and $d_w > \max\{\dim E + 1, d_u\}$ then system (3.1), (3.2) is generically strongly differentially observable of order k and the stationary set E is generically asymptotically stabilizable by (3.1), (3.2).

Special case: $E = \{q_0\}$ **Aeyels** (1985), **Gauthier, Kupka** (1994).

Remark 3.1 Modifications of the observer (3.2).

a) High-gain extended *Kalman filter*, where K_Θ is *not constant*;
b) High-gain observer where the observations are *sampled*;
c) Observer for *joint state* and *paramter estimation*.
d) We avoid the use of derivatives of the measurements