Time series analysis of elasto-plastic bifurcations based on extremely short observation times

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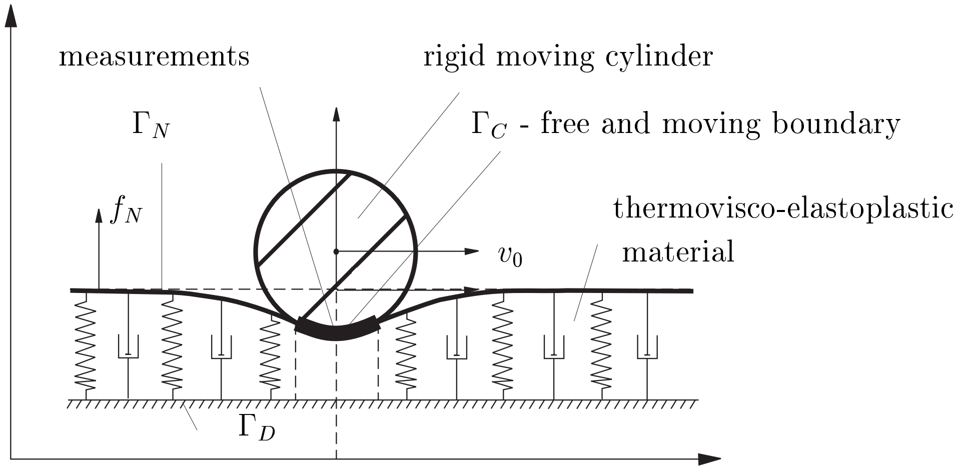
1 Thermovisco-elastoplastic contact

1.1 The mechanical model



1.2 Notation

Suppose $\Omega \subset \mathbb{R}^m$ is a domain (reference configuration of the visco-elastoplastic body), $\Gamma = \partial\Omega$ is the piecewise Lipschitz continuous boundary divided in the three disjunct parts Γ_D (where the body is clamped), Γ_N (where the tractions act) and Γ_C (where the visco-elastoplastic body comes in frictional contact with a rigid moving body).
Assume that $x = (x^1, \ldots, x^m)$ is the location in $\Omega, t \in \mathbb{R}_+$ is the time, $n = (n^1, \ldots, n^m)$ is the unit normal to $\Gamma, u(x,t) = (u^1(x,t), \ldots, u^m(x,t))$ are the displacements, $\Theta = \Theta(x,t)$ is the temperature, $\sigma = (\sigma^{ij})$ is the stress tensor, $f_A = (f_A^1(x,t), \ldots, f_A^m(x,t))$ are the body forces in Ω.

1.3 Elastoplastic and heat equations

The *equations of motion* and *heat transfer* are given by

$$[\sigma^{kj}(\delta_k^i + u^i_{,k})]_{,j} + f_A^i = \ddot{u}^i \qquad \text{in } \Omega \times (0,T)\,, \tag{1.1}$$

$$\dot{\Theta} - (k^{ij}\Theta_{,j})_{,i} = -c^{ij}u_{i,j} + q_V \quad \text{in } \Omega \times (0,T)\,, \tag{1.2}$$

where $u_{i,j} = \frac{\partial u_i}{\partial x^j} - \Gamma^k_{ji}u_k$ is the j-th covariant derivative of u_i computed with respect to the Christoffel symbol $\Gamma^k_{ji}(x)$ in Ω (repeated indices indicate summation from 1 to m), $u^i_{,k}$ is the k-th covariant derivative of $u^i(u_i$ and u^i are the covariant and contravariant components of u, respectively), δ^i_k is the Kronecker symbol, $\Theta_{,j}$ denotes the j-th covariant derivative, q_V is the density of volume heat sources, $c^{ij} = c^{ij}(x)$ and $k^{ij} = k^{ij}(x)$ are the tensors of thermal expansion and thermal conductivity, respectively. The stress tensor $\sigma = (\sigma^{ij})$ is defined by the *thermovisco-elastoplastic stress-strain relation*

$$\sigma^{ij} = a^{ijkl} u_{k,l} + b^{ijkl} \dot{u}_{k,l} - c^{ij}\Theta + \mathcal{P}^{ij}[u_{k,l}, \Theta] \quad \text{in } \Omega \times (0,T)\,, \tag{1.3}$$

where $a = (a^{ijkl})$ and $b = (b^{ijkl})$ are the tensors of elastic and viscosity coefficients, respectively, $\{\mathcal{P}^{ij}[\cdot, \Theta]\}_{\Theta > 0}$ is the plastic part given by Θ-dependent hysteresis operators.

As *boundary and initial conditions* we have ([4]):

a) *Prescribed displacements and temperature*

$$\begin{aligned} u &= 0 \quad \text{on} \quad \Gamma_D \times (0,T)\,; \\ \Theta &= \Theta_b \ \text{ on } \quad (\Gamma_D \cup \Gamma_N) \times (0,T)\,; \\ u(\cdot,0) &= u_0\,,\ \dot{u}(\cdot,0) = u_1,\ \Theta(\cdot,0) = \Theta_0 \quad \text{in}\,\Omega\,; \end{aligned} \tag{1.4}$$

b) *Prescribed boundary forces*

$$\sigma^{ij} n_j = f_N^i \quad \text{on} \quad \Gamma_N \times (0,T)\,, \tag{1.5}$$

where $f_N = (f_N^i(x,t))$ are the applied tractions;

c) *Frictional stress and temperature on* Γ_C

By Coulomb's law of dry friction

$$\begin{aligned} &|\sigma_{\mathcal{T}}| \le \mu|\sigma_N|(1 - \delta_W|\sigma_N|)_+ \qquad \text{on } \Gamma_C \times (0,T)\,, \\ &|\sigma_{\mathcal{T}}| < \mu|\sigma_N|(1 - \delta_W|\sigma_N|)_+ \ \Rightarrow\ \dot{u}_{\mathcal{T}} = v_0 \quad \text{(stick zone)}\,, \\ &|\sigma_{\mathcal{T}}| = \mu|\sigma_N|(1 - \delta_W|\sigma_N|)_+ \ \Rightarrow\ \dot{u}_{\mathcal{T}} = \ v_0 - \lambda_S \sigma_{\mathcal{T}} \quad \text{(slip zone)}\,, \end{aligned} \tag{1.6}$$

$$k^{ij}\Theta_{,i} n_j = \mu|\sigma_N|(1 - \delta_W|\sigma_N|)_+ s_C(\cdot, |\dot{u}_{\mathcal{T}} - v_0|) - k_e(\Theta - \Theta_R)\,, \tag{1.7}$$

where $\sigma_N = \sigma^{ij} n_i n_j$ and $u_N = u^i n_i$ are the normal components of σ and u on Γ, respectively, $\sigma_{\mathcal{T}}^i = \sigma^{ij} n_j - \sigma_N n^i$ and $u_{\mathcal{T}}^i = u^i - u_N n^i$ are the tangential components of σ and u on Γ, respectively, μ is the friction coefficient, v_0 is the velocity of the moving rigid body, $\lambda_S \ge 0$ is a multiplier indicating the direction of the slip, δ_W is a positive constant related to wear, Θ_R is the temperature of the rigid body, $s_C(\cdot, r)$ is a prescribed distance function and k_e is coefficient of heat exchange between elastoplastic body and rigid body.

In general there are no classical solutions for (1.1) – (1.7).

2 Coupled variational systems

2.1 Scales of Hilbert spaces

A collection of real Hilbert spaces $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ with norm $\|\cdot\|_\alpha$ and scalar product $(\cdot,\cdot)_\alpha$ is called *scale* of Hilbert spaces if the following is true:

(i) For any $\alpha > \beta$ the space H_α is continuously embedded into H_β, i.e. $H_\alpha \subset H_\beta$ and there exists a $c_1 > 0$ such that $\|h\|_\beta \le c_1\|h\|_\alpha$, $\forall\, h \in H_\alpha$, and H_α is dense in H_β ;

(ii) For any $\alpha > 0$ and $h \in H_\alpha$ the linear functional $(\cdot, h)_0$ on H_0 can be continuously extended to a linear continuous functional $(\cdot, h)_{-\alpha,\alpha}$ on $H_{-\alpha}$ satisfying $|(h', h)_{-\alpha,\alpha}| \le \|h'\|_{-\alpha}\|h\|_\alpha$, $\forall\, h' \in H_{-\alpha}$, $\forall\, h \in H_\alpha$. Any linear continuous functional l on H_α has the form $l(h) = (h', h)_{-\alpha,\alpha}$ with some $h' \in H_{-\alpha}$, i.e., $H_{-\alpha}$ is isomorphic to the space of linear continuous functionals on H_α.

From **(i)** it follows that for any $\alpha \in (\beta, \gamma)$ the space H_α is *rigged* by H_β and H_γ, i.e., $H_\gamma \subset H_\alpha \subset H_\beta$ with dense and continuous embeddings.

Suppose that $\tilde{H}_1 \subset \tilde{H}_0$ are densely and continuously embedded Hilbert spaces and $a : \tilde{H}_1 \times \tilde{H}_1 \to \mathbb{R}$ is a continuous bilinear form, i.e., there exists a $c_2 > 0$ such that $a(h, h') \le c_2\|h\|_1\|h'\|_1$, $\forall\, h, h' \in \tilde{H}_1$. Then there exists a scale of Hilbert spaces $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ with $H_1 = \tilde{H}_1, H_0 = \tilde{H}_0$ and a linear bounded operator $A : H_1 \to H_{-1}$ such that

$$(Ah, h')_{-1,1} = a(h, h')\,,\ \forall\, h, h' \in H_1.$$

Example 2.1 Suppose $\Omega \subset \mathbb{R}^m$ is a domain and N is an arbitrary natural number. $\{H_\alpha^{(N)}\}_{\alpha\in\mathbb{R}}$ is the *scale* of *fractional Sobolev spaces* such that $H_l^{(N)} = W^{l,2}(\Omega)$,
$l = 0, 1, \ldots , N$, with norms $\|u\|^2_{H_\alpha^{(N)}}$ given by

$$\int_\Omega (|u|^2 + \sum_{|\beta|=1}^{\alpha} |D^\beta u|^2)dx =: \|u\|^2_{W^{\alpha,2}}, \quad \text{if} \quad \alpha \ge 0 \text{ integer},$$

$$\|u\|^2_{W^{k,2}} + \sum_{|\beta|=k} \int_\Omega\int_\Omega \frac{|D^\beta u(x) - D^\beta u(y)|^2}{|x-y|^{k+2\lambda}}dxdy, \text{ if } \alpha = k + \lambda > 0\,, k \ge 0 \text{ integer}, \ \lambda \in (0, 1),$$

$$\sup_{\|v\|_{H_{-\alpha}^{(N)}}=1} |\int_\Omega u(x)v(x)dx|\,, \text{if} \quad \alpha < 0.$$

□

2.2 A simplified contact problem

Suppose $\Omega \subset \mathbb{R}^m$ is a bounded domain, $\partial\,\Omega$ is smooth, $u = u(x,t)$ and $\Theta = \Theta(x,t)$ are the displacement and the temperature in the elastic body satisfying the system

$$\begin{aligned} u_{tt} + 2\varepsilon u_t - \Delta u + \alpha u &= \xi(t), \quad \xi(t) \in \varphi(\Theta(t))\,, & (2.1)\\ \Theta_t - \beta\Delta\Theta + u - \gamma\zeta(t) &= 0, \quad \zeta(t) = g(\Theta(t))\,, & (2.2)\end{aligned}$$

with $\alpha, \beta, \varepsilon, \gamma$ constants, and the boundary and initial conditions

$$u = 0\,,\ \Theta = 0 \quad \text{on } \partial\,\Omega \times (0,T) \qquad (2.3)$$

$$u(\cdot,0) = u_0(\cdot)\,,\ \dot{u}(\cdot,0) = u_1(\cdot)\,, \Theta(\cdot,0) = \Theta_0 \quad \text{in } \Omega. \qquad (2.4)$$

Assume that $\varphi : \mathbb{R} \to 2^{\mathbb{R}}$ and g : $\mathbb{R} \to \mathbb{R}$ are nonlinear maps satisfying

$$vg(v) - \xi^2 \geq 0,\ \forall v \in \mathbb{R},\ \forall \xi \in \varphi(v) \qquad (2.5)$$

and $g = \phi'$, i.e. g has a Fréchet differentiable potential.
Suppose A_0 is the self-adjoint positive-definite operator generated by $(-\Delta)$ with zero boundary conditions and having the domain $\mathcal{D}(A_0) = W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$. Introduce the spaces $\mathcal{V}_0 = L^2(\Omega), \mathcal{V}_1 = \mathcal{D}(A_0^{1/2})$ and $\mathcal{V}_2 = \mathcal{D}(A_0)$ with

$$(u,v)_s = (A_0^{s/2} u, A_0^{s/2} v)\ ,\ \forall u, v \in \mathcal{V}_s, s = 0,1,2\,, \qquad (2.6)$$

as scalar product and $Y_s = \mathcal{V}_{s+1} \times \mathcal{V}_s\,,\ Z_s = \mathcal{V}_{s+1}\,, s = 0,1\,,$ with the scalar product in Y_s given by

$$((u,v),(\bar{u},\bar{v}))_s = (u,\bar{u})_{s+1} + (v,\bar{v})_s\,, \quad \forall (u,v),(\bar{u},\bar{v}) \in Y_s\,. \qquad (2.7)$$

3 Observations for elastoplastic bifurcations

The weak form of (2.1), (2.2) is a *parameter-dependent hybrid system consisting* of a *variational inequality* and a *variational equality* of the type

$$(\dot{y} - A(q)y - B(q)\xi, \eta - y)_{Y_{-1},Y_1} + \ \Psi(\eta,q) - \Psi(y,q) \geq 0\,, \qquad (3.1)$$

$$w(t) = C(q)y, \qquad \xi(t) \in \varphi(t, w(t), v(t), q)\,, \qquad (3.2)$$
$\forall \eta \in L^2(0,T;Y_1),$ a.a. on $(0,T)$,

$$(\dot{z} - A_1(q)z - B_1(q)\zeta, \vartheta)_{Z_{-1},Z_1} = 0\,, \qquad (3.3)$$

$$v(t) = C_1(q)z\,, \qquad \zeta(t) \in g(t, w(t), v(t), q)\,,$$
$\forall \vartheta \in L^2(0,T;Z_1),$ a.a. on $(0,T)\,.$ (3.4)

Here $q \in Q$ is a parameter, (Q,d) is a metric space. For any $q \in Q$ we assume that $A(q) \in \mathcal{L}(Y_1, Y_{-1}), B(q) \in \mathcal{L}(\Xi, Y_{-1}),\ C(q) \in \mathcal{L}(Y_{-1}, W), A_1(q) \in \mathcal{L}(Z_1, Z_{-1}), B_1(q) \in \mathcal{L}(\mathcal{Z}, Z_{-1}), C_1(q) \in \mathcal{L}(Z_{-1}, \Upsilon)$ are linear maps. Assume also that

$$
\begin{aligned}
&\Psi(\cdot, q) : Y_1 \to \mathbb{R}_+, \quad \varphi(\cdot,\cdot,\cdot,q) : \mathbb{R}_+ \times W \times \Upsilon \to 2^{\Xi},\\
&g(\cdot,\cdot,\cdot,q) : \mathbb{R}_+ \times W \times \Upsilon \to \mathcal{Z},
\end{aligned}
$$

are nonlinear set-valued and point maps, respectively, $Y_1, Y_{-1}, Z_1, Z_{-1}, \Xi, W, \mathcal{Z}, \Upsilon$ are real Hilbert spaces. A pair $\{y(\cdot), z(\cdot)\} \in L^2(0,T;Y_1) \times L^2(0,T;Z_1)$ is said to be a *solution* of (3.1) – (3.4) on $(0,T)$ if $\{\dot y(\cdot), \dot z(\cdot)\} \in L^2(0,T;Y_{-1}) \times L^2(0,T;Z_{-1})$ and there exists a pair $\{\xi(\cdot), \zeta(\cdot)\} \in L^2(0,T;\Xi) \times L^2(0,T;\mathcal{Z})$ such that $\{y(\cdot), z(\cdot), \xi(\cdot), \zeta(\cdot)\}$ satisfies (3.1) – (3.4) for a.a. $t \in (0,T)$ and $\int_0^T \Psi(y(t),q)dt < +\infty$. We assume that for any $T > 0$ such solutions exist.

Definition 3.1 *Suppose that $\{S_\alpha\}, \{\tilde S_\alpha\}, \{R_\alpha\}$ and $\{\tilde R_\alpha\}$ are scales of real Hilbert spaces (observation and output spaces, respectively) and $D_\alpha \in \mathcal{L}(Y_1, S_\alpha), E_\alpha \in \mathcal{L}(\Xi, S_\alpha), \tilde D_\alpha \in \mathcal{L}(Z_1, \tilde S_\alpha), \tilde E_\alpha \in \mathcal{L}(\mathcal{Z}, \tilde R_\alpha), M_\alpha \in \mathcal{L}(Y_1, R_\alpha), N_\alpha \in \mathcal{L}(\Xi, R_\alpha), \tilde M_\alpha \in \mathcal{L}(Z_1, \tilde R_\alpha)$ and $\tilde N_\alpha \in \mathcal{L}(\mathcal{Z}, \tilde R_\alpha)$ are scales of linear operators (observation and output operators, respectively).*
If $\{y(\cdot), z(\cdot), \xi(\cdot), \zeta(\cdot)\}$ is a response of (3.1) – (3.4) and $\alpha, \tilde\alpha, \beta, \tilde\beta \in \mathbb{R}$ are arbitrary scale parameters, the function

$$
s(\cdot, \alpha, \tilde\alpha) = (D_\alpha y(\cdot) + E_\alpha \xi(\cdot), \tilde D_{\tilde\alpha} z(\cdot) + \tilde E_{\tilde\alpha} \zeta(\cdot)) \tag{3.5}
$$

is called observation (measurement or time series) and the function

$$
r(\cdot, \beta, \tilde\beta) = (M_\beta y(\cdot) + N_\beta \xi(\cdot), \tilde M_{\tilde\beta} z(\cdot) + \tilde N_{\tilde\beta} \zeta(\cdot)) , \tag{3.6}
$$

is called (unobservable) output of (3.1) – (3.4). For two responses

$$
\{y_i(\cdot), z_i(\cdot), \xi_i(\cdot), \zeta_i(\cdot)\} , \quad i = 1, 2 , \tag{3.7}
$$

of (3.1) – (3.4) and arbitrary scale parameters $\alpha, \tilde\alpha, \beta, \tilde\beta \in \mathbb{R}$ we define the deviations

$$
\begin{aligned}
&\Delta y(\cdot) = y_1(\cdot) - y_2(\cdot), \quad \Delta\, z(\cdot) = z_1(\cdot) - z_2(\cdot),\\
&\Delta\, \xi(\cdot) = \xi_1(\cdot) - \xi_2(\cdot), \quad \Delta\, \zeta(\cdot) = \zeta_1(\cdot) - \zeta_2(\cdot) ,
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
\Delta\, s(\cdot, \alpha)^2 &= \|D_\alpha \Delta\, y(\cdot) + E_\alpha \Delta\, \xi(\cdot)\|^2_{S_\alpha} ,\\
\Delta\, \tilde s(\cdot, \tilde\alpha)^2 &= \|\tilde D_{\tilde\alpha} \Delta\, z(\cdot) + \tilde E_{\tilde\alpha} \Delta\, \zeta(\cdot)\|^2_{\tilde S_{\tilde\alpha}} ,
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
\Delta\, r(\cdot, \beta)^2 &= \|M_\beta \Delta\, y(\cdot) + N_\beta \Delta\, \xi(\cdot)\|^2_{R_\beta} ,\\
\Delta \tilde r(\cdot, \tilde\beta)^2 &= \|\tilde M_{\tilde\beta} \Delta z(\cdot) + \tilde N_{\tilde\beta} \Delta \zeta(\cdot)\|^2_{\tilde R_{\tilde\beta}} .
\end{aligned} \tag{3.10}
$$

Definition 3.2 *Suppose that $a > 0, b > 0 (a < b)$ and $t_1 > 0$ are numbers. The observation (3.5) is determining for the bifurcation "loss of (a, b, t_1)-stability" of the output (3.6) at $q = q^*$ if there exist continuous near q^* real-valued functions $\alpha(\cdot), \tilde{\alpha}(\cdot), \beta(\cdot)$ and $\tilde{\beta}(\cdot)$ with the following properties:*

a) For $q = q_1$ the observation (3.5) with $\alpha = \alpha(q_1), \tilde{\alpha} = \tilde{\alpha}(q_1)$ is determining for the (a, b, t_1)-stability of the output (3.6) with $\beta = \beta(q_1), \tilde{\beta} = \tilde{\beta}(q_1)$, i.e., there exists an $\varepsilon_1 = \varepsilon_1(q_1) > 0$ such that for arbitrary two responses (3.7) and their deviations (3.8) – (3.10) which satisfy

$$\Delta\, r(0, \beta(q_1))^2 + \Delta\, \tilde{r}(0, \tilde{\beta}(q_1))^2 < a \tag{3.11}$$

the observation property

$$\int_0^{t_1} [\Delta\, s(t, \alpha(q_1))^2 + \Delta\, \tilde{s}(t, \tilde{\alpha}(q_1))^2] dt < \varepsilon_1 \tag{3.12}$$

implies the output property

$$\Delta\, r(t, \beta(q_1))^2 + \Delta\, \tilde{r}(t, \tilde{\beta}(q_1))^2 < b\,,\ \forall t \in (0, t_1)\,.$$

b) For $q = q_2$ the observation (3.5) with $\alpha = \alpha(q_2), \tilde{\alpha} = \tilde{\alpha}(q_2)$ is determining for the (a, b, t_1)-instability of the output (3.6) with $\beta = \beta(q_2), \tilde{\beta} = \tilde{\beta}(q_2)$, i.e., there exists an $\varepsilon_2 = \varepsilon_2(q_2) > 0$ such that for arbitrary two responses (3.7) and their deviations (3.8) – (3.10) which satisfy (3.11) the observation property

$$\int_0^{t^*} [\Delta\, s(t, \alpha(q_2))^2 + \Delta\, \tilde{s}(t, \tilde{\alpha}(q_2))^2] dt \geq \varepsilon_2$$

for a time $t^* \in (0, t_1)$ implies the output property

$$\Delta\, r(t^*, \beta(q_2))^2 + \Delta\, \tilde{r}(t^*, \tilde{\beta}(q_2))^2 \geq b\,.$$

Remark 3.1 The "loss of (a, b, t_1)-stability"-bifurcation for visco-elastoplastic systems (3.1) – (3.4) means the loss of stability on a finite time interval and is connected with the creation of almost-periodic solutions ([3]). Frequency-domain conditions for observations of this type of bifurcation are derived in [1]. Oberservations that are determining for upper fractal dimension estimates of negatively invariant sets of variational inequalities are considered in [2]. □

Definition 3.3 *Suppose that $q \in Q$ is arbitrary and $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in \mathbb{R}, a > 0$ are arbitrary numbers. The observation (3.5) is determining for the a-convergence of the output (3.6) if for any two responses (3.7) of (3.1) – (3.4) and their deviations (3.8) – (3.10) from*

$$\int_t^{t+1} [\Delta\, s(\tau, \alpha)^2 + \Delta\, \tilde{s}(\tau, \tilde{\alpha})^2] d\tau \to 0 \tag{3.13}$$

for $t \to +\infty$ it follows that

$$\limsup_{t\to+\infty}[\Delta\, r(t,\beta)^2 + \Delta\tilde{r}(t,\tilde{\beta})^2] \le a\,. \tag{3.14}$$

Remark 3.2 The property (3.13), (3.14) means that observations over *sequences of short time-intervals* in *one point* of the elastoplastic body may be sufficient in order to get information about the asymptotic behaviour of the *full solution.* □

4 Frequency-domain conditions for determining observations

4.1 Description of the uncertainty nonlinear part

Consider the system (3.1) – (3.4) with arbitrary but fixed $q \in Q$. Suppose that $F(\cdot,\cdot,q)$ and $G(\cdot,\cdot,q)$ are quadratic forms on $Y_1 \times \Xi$. The *class* $\mathfrak{N}(F,G)$ *of nonlinearities* for (3.1) consists of all set-valued maps

$$\varphi(\cdot,\cdot,\cdot,q) : \mathbb{R}_+ \times W \times \Upsilon \to 2^{\Xi} \tag{4.1}$$

satisfying the following property: For any sufficiently large $t_0, T, 0 < t_0 < T$, and any pairs of functions $y_1(\cdot), y_2(\cdot) \in L^2(0,T;Y_1)$, $z_1(\cdot), z_2(\cdot) \in L^2(0,T;Z_1)$ and $\xi_1(\cdot), \xi_2(\cdot) \in L^2(0,T;\Xi)$ with

$$\xi_i(t) \in \varphi(t, C(q)y_i(t), C_1(q)z_i(t), q), \quad i=1,2, \quad \text{a.a. } t \in [0,T], \tag{4.2}$$

$$\text{and} \qquad \|C_1(q)z_i(t)\|_{\Upsilon} \le \Delta, \quad i=1,2, \quad \text{a.a. } t \in [t_0,T], \tag{4.3}$$

where $\Delta > 0$ is a small number depending on the second subsystem (3.3), (3.4), it follows that

$$F(y_1(t)-y_2(t), \xi_1(t)-\xi_2(t), q) \ge 0 \quad \text{a.a. } t \in [t_0,T]\,, \tag{4.4}$$

and there exist a continuous function $\Phi : W \to \mathbb{R}$ (*generalized potential*) and numbers $\lambda = \lambda(q) > 0$ and $\gamma = \gamma(q) > 0$ such that

$$\begin{aligned}
&\int_s^t G(y_1(\tau)-y_2(\tau), \xi_1(\tau)-\xi_2(\tau), q)d\tau \\
&\ge \frac{1}{2}[\Phi(C(q)y_1(t)-C(q)y_2(t)) - \Phi(C(q)y_1(s)-C(q)y_2(s))] \\
&+\lambda\int_s^t \Phi(C(q)y_1(\tau)-C(q)y_2(\tau))d\tau \quad \text{for all} \quad s,t \in [t_0,T]\,, s \le t,
\end{aligned}$$

and

$$\Phi(C(q)y_1(t)-C(q)y_2(t)) \ge \gamma\|C(q)y_1(t)-C(q)y_2(t)\|_W^2, \quad \text{a.a.} \quad t \in [t_0,T]. \tag{4.5}$$

4.2 Assumptions for the existence of determining observers

If $T > 0$ is an arbitrary number we define the norm for Bochner measurable functions in $L^2(0,T;Y_j)$, $j = 1, 0, -1$, through $\|y(\cdot)\|_{2,j} = (\int_0^T \|y(t)\|_j^2 dt)^{1/2}$. Let $\mathfrak{W}_T$ be the space of functions $y(\cdot) \in L^2(0,T;Y_1)$ for which $\dot{y}(\cdot) \in L^2(0,T;Y_{-1})$ equipped with the norm

$$\|y(\cdot)\|_{\mathfrak{W}_T} = (\|y(\cdot)\|_{2,1}^2 + \|\dot{y}(\cdot)\|_{2,-1}^2)^{1/2} \tag{4.6}$$

(**A1**) There exists a number $\lambda = \lambda(q) > 0$ such that for any $T > 0$ and any element $f \in L^2(0,T;Y_{-1})$ the problem

$$\dot{y} = (A(q) + \lambda I)y + f(t), y(0) = y_0, \tag{4.7}$$

is *well-posed*, i.e., for arbitrary $y_0 \in Y_0, f(\cdot) \in L^2(0,T;Y_{-1})$ there exists a unique solution $y(\cdot) \in \mathfrak{W}_T$ satisfying (4.8) in a variational sense and depending continuously on the initial data, i.e., $\|y(\cdot)\|_{\mathfrak{W}_T}^2 \le c_1\|y_0\|_0^2 + c_2\|f(\cdot)\|_{2,-1}^2$, where $c_1 > 0$ and $c_2 > 0$ are some constants. Furthermore, any solution of $\dot{y} = (A(q) + \lambda I)y, y(0) = y_0$, is exponentially decreasing for $t \to +\infty$, i.e., there exist constants $c_3 > 0$ and $\varepsilon > 0$ such that $\|y(t)\|_0 \le c_3 e^{-\varepsilon t}\|y_0\|_0, t > 0$.
(**A2**) There exists a number $\lambda = \lambda(q) > 0$ such that the operator $A(q) + \lambda I \in \mathcal{L}(Y_1, Y_{-1})$ is *regular*, i.e., for any $T > 0, y_0 \in Y_1, z_T \in Y_1$ and $f \in L^2(0,T;Y_0)$ the solutions of the *direct problem* $\dot{y} = (A(q) + \lambda I)y + f(t), y(0) = y_0$, and of the associated *dual problem* $\dot{z} = -(A(q) + \lambda I)^* z + f(t), z(T) = z_T$, are strongly continuous in t in the norm of Y_1.
(**A3**) There exist numbers $\lambda = \lambda(q) > 0, \delta = \delta(q) > 0$ and $\alpha = \alpha(q)$ such that the following two properties hold:

$$\begin{aligned} &\text{a)}\quad F^c(y,\xi,q) + G^c(y,\xi,q) - \delta\|D_\alpha^c y + E_\alpha^c \xi\|_{S_\alpha^c}^2 \le 0\,, \\ &\forall (y,\xi) \in Y_1^c \times \Xi^c \exists\, \omega \in \mathbb{R} : i\omega y = (A^c(q) + \lambda I^c)y + B^c(q)\xi\,; \end{aligned} \tag{4.8}$$

b) The functional

$$J(y(\cdot),\xi(\cdot)) = \int_0^\infty [F^c(y(\tau),\xi(\tau),q) + G^c(y(\tau),\xi(\tau),q) -$$

$$\delta\|D_\alpha^c y(\tau) + E_\alpha^c \xi(\tau)\|_{S_\alpha^c}^2]\, d\tau$$

is bounded from above on the set

$$\mathfrak{M}_{y_0} = \{y(\cdot),\xi(\cdot) : \dot{y} = (A^c(q) + \lambda I^c)y + B^c(q)\xi,$$

$$y(0) = y_0, y(\cdot) \in \mathfrak{W}_\infty^c\,,\ \xi(\cdot) \in L^2(0,\infty;\Xi^c\}\quad \text{for any}\quad y_0 \in Y_0^c.$$

Here $F^c, G^c, D_\alpha^c, E_\alpha^c, A^c, I^c, B^c, S_\alpha^c, \mathfrak{W}_\infty^c, \Xi^c$ denote the usual complexification of quadratic forms, linear operators and Hilbert spaces, respectively.

Theorem 4.1 *Suppose that there exist numbers $\lambda = \lambda(q) > 0, \delta = \delta(q) > 0$ and $\alpha = \alpha(q)$ such that the assumptions (**A1**) – (**A3**)are satisfied. Suppose also that for any solutions of (3.1) – (3.4) there are a time $t_0 > 0$ and a number $\Delta > 0$ such that (4.3) is fulfilled for any $T > t_0$. Then the observation*

$$s(\cdot) = (D_\alpha y(\cdot) + E_\alpha \xi(\cdot), 0) \tag{4.9}$$

is determining for the output a-convergence in (3.1), (3.4) with respect to the output

$$r(\cdot) = w(\cdot) = C(q)y(\cdot)\,, \tag{4.10}$$

where $a > 0$ is a certain number depending on $\Psi(\cdot, q)$ in (3.1).

4.3 Completeness defect of the observation operators

The frequency-domain condition (**A3**) depends on embedding properties of the Sobolev spaces under consideration. Assume, for example, that $G \equiv 0, E_\alpha = 0$ and $F(y, \xi, q) = q_1\|y\|_0^2 - q_2\|y\|_1^2, (y, \xi) \in Y_0 \times \Xi$, where q_1 and q_2 are certain real constants and $q = (q_1, q_2, \tilde{q}) \in Q$. In order to verify (4.4) we introduce the frequency-domain characteristic $\chi(i\omega, q) = (i\omega I^c - A^c_\lambda(q))^{-1}B^c(q)$ for $\omega \in \mathbb{R}$ s.t. $i\omega \in \rho(A^c_\lambda(q))$, where $A^c_\lambda(q) = A^c(q) + \lambda I^c$. The frequency-domain condition (4.4) is satisfied if

$$\begin{aligned} &q_1\|\chi(i\omega, q)\xi\|^2_{Y_0^c} - q_2\|\chi(i\omega, q)\xi\|^2_{Y_1^c} - \delta\|D^c_\alpha\chi(i\omega, q)\xi\|^2_{S^c_\alpha} \le 0, \\ &\forall \xi \in \Xi^c, \forall\omega \in \mathbb{R} : i\omega \in \rho(A^c_\lambda(q)). \end{aligned} \tag{4.11}$$

Suppose that from the embedding $Y_1^c \subset Y_0^c \subset Y_{-1}^c$ and the properties of D_α we have the a priori estimate

$$\|v\|^2_{Y_0^c} \le c_1\|v\|^2_{Y_1^c} + c_2\varepsilon_{D^c_\alpha}\|D^c_\alpha v\|^2_{S^c_\alpha}\ ,\ \forall v \in Y_1^c, \tag{4.12}$$

where $c_1 > 0$ and $c_2 > 0$ are certain constants and

$$\varepsilon_{D^c_\alpha} = \varepsilon_{D^c_\alpha}(Y_1^c, Y_0^c) = \sup\{\|w\|_{Y_0^c} : w \in Y_1^c, D^c_\alpha w = 0\,, \|w\|_{Y_1^c} \le 1\}$$

is the *completeness* defect of the observation operator D^c_α with respect to the embedding $Y_1^c \subset Y_0^c$. It follows from (4.13) that the frequency-domain condition (4.12) is satisfied if

$$\begin{aligned} &q_1c_1\|\chi(i\omega, q)\xi\|^2_{Y_1^c} - q_2\|\chi(i\omega, q)\xi\|^2_{Y_1^c} + q_1c_2\varepsilon_{D^c_\alpha}\|D^c_\alpha\chi(i\omega, q)\xi\|^2_{S^c_\alpha} - \\ &\delta\|D^c_\alpha\chi(i\omega, q)\xi\|^2_{S^c_\alpha} \le 0, \quad \forall\xi \in \Xi^c, \quad \forall\omega \in \mathbb{R} : i\omega \in \rho\left(A^c_\lambda(q)\right). \end{aligned} \tag{4.13}$$

For (4.12) it is sufficient that

$$q_1c_1 - q_2 \le 0 \quad \text{and} \quad q_1c_2\varepsilon_{D^c_\alpha} - \delta \le 0\,. \tag{4.14}$$

The inequalities (4.14) describe a subset in the space of parameters of the variational inequality and of the observation operator. The second condition from (4.15) is always satisfied if $\varepsilon_{D^c_{\tilde\alpha}}$ is sufficiently small. Suppose that $D_\alpha y = (l_1(y), \dots, l_k(y))$, where $l_i : Y_1 \to \mathbb{R}$, $i = 1, \dots, k$, are continuous linear functionals and $Y_1 = W^{s,2}(\Omega), Y_0 = W^{\sigma,2}(\Omega)$ with $s > \sigma$. Then $\varepsilon_{D^c_{\tilde\alpha}} \approx c_1(\frac{c_2}{k})^{s-\sigma}$, i.e., the completeness defect of the observation operator D_α depends on the smoothness properties of the embedding $Y_1^c \subset Y_0^c$.

5 Frequency-domain conditions for observation stability

Let us consider the hybrid system (3.1) – (3.4) with $\Psi \equiv 0$ as a first order variational equation with a set-valued nonlinearity. For this we define the new variables

$$\mathbf{y} = (y, z), \quad \mathbf{w} = (w, z), \quad \boldsymbol{\xi} = (\xi, \zeta), \quad \boldsymbol{\eta} = (\eta, \vartheta), \tag{5.1}$$

the product spaces

$$\mathcal{Y}_i = Y_i \times Z_i\,,\ i = 1, 0, -1, \quad \mathcal{W} = W \times \Upsilon, \quad \mathcal{U} = \Xi \times \mathcal{Z}\,, \tag{5.2}$$

the parameter-dependent operator matrices

$$\mathcal{A}(q) = \begin{bmatrix} A(q) & 0 \\ 0 & A_1(q) \end{bmatrix}, \quad \mathcal{B}(q) = \begin{bmatrix} B(q) \\ B_1(q) \end{bmatrix}, \quad \mathcal{C}(q) = [C(q), C_1(q)]\,, \tag{5.3}$$

and the nonlinear set-valued map

$$\boldsymbol{\varphi}(\cdot,\cdot,q) = (\varphi(\cdot,\cdot,\cdot,q), \quad g(\cdot,\cdot,\cdot,q)) : \mathbb{R}_+ \times \mathcal{W} \to 2^{\Xi} \times \mathcal{Z}\,. \tag{5.4}$$

Thus we can write the coupled system (3.1) – (3.4) as first order variational equation with set-valued nonlinearity in $\mathcal{Y}_{-1}$ as

$$\dot{\mathbf{y}} = \mathcal{A}(q)\mathbf{y} + \mathcal{B}(q)\boldsymbol{\xi}\,, \tag{5.5}$$

$$\mathbf{w}(t) = \mathcal{C}(q)\mathbf{y}(t), \quad \boldsymbol{\xi}(t) \in \boldsymbol{\varphi}\,(t, \mathbf{w}(t), q). \tag{5.6}$$

The scales of observation resp. output spaces for (5.5), (5.6) are

$$\mathcal{S}_{\boldsymbol{\alpha}} = S_\alpha \times \tilde{S}_{\tilde\alpha}, \quad \mathcal{R}_{\boldsymbol{\alpha}} = R_\alpha \times \tilde{R}_{\tilde\alpha}, \quad \boldsymbol{\alpha} = (\alpha, \tilde\alpha) \in \mathbb{R}^2\,, \tag{5.7}$$

the scales of observation resp. output operators are

$$\mathcal{D}_{\boldsymbol{\alpha}} = \begin{bmatrix} D_\alpha & 0 \\ 0 & \tilde{D}_{\tilde\alpha} \end{bmatrix}, \quad \mathcal{E}_{\boldsymbol{\alpha}} = \begin{bmatrix} E_\alpha & 0 \\ 0 & \tilde{E}_{\tilde\alpha} \end{bmatrix}, \quad \mathcal{M}_{\boldsymbol{\alpha}} = \begin{bmatrix} M_\alpha & 0 \\ 0 & \tilde{M}_{\tilde\alpha} \end{bmatrix},$$

$$\mathcal{N}_{\boldsymbol{\alpha}} = \begin{bmatrix} N_\alpha & 0 \\ 0 & \tilde{N}_{\tilde{\alpha}} \end{bmatrix} . \tag{5.8}$$

It is clear that

$$\mathcal{D}_{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{Y}_1, \mathcal{S}_{\boldsymbol{\alpha}}), \quad \mathcal{E}_{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{U}, \mathcal{S}_{\boldsymbol{\alpha}}), \quad \mathcal{M}_{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{Y}_1, \mathcal{R}_{\boldsymbol{\alpha}}), \quad \mathcal{N}_{\boldsymbol{\alpha}} \in \mathcal{L}(\mathcal{U}, \mathcal{R}_{\boldsymbol{\alpha}}), \quad \boldsymbol{\alpha} \in \mathbb{R}^2 . \tag{5.9}$$

If $\{\mathbf{y}(\cdot), \boldsymbol{\xi}(\cdot)\}$ is a response of (5.5), (5.6) and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^2$ are arbitrary scale parameters the function

$$\mathbf{s}(\cdot, \boldsymbol{\alpha}) = \mathcal{D}_{\boldsymbol{\alpha}} \mathbf{y}(\cdot) + \mathcal{E}_{\boldsymbol{\alpha}} \boldsymbol{\xi}\,(\cdot) \tag{5.10}$$

is the observation and

$$\mathbf{r}(\cdot, \boldsymbol{\beta}) = \mathcal{M}_{\beta} \mathbf{y}(\cdot) + \mathcal{N}_{\beta} \boldsymbol{\xi}(\cdot) \tag{5.11}$$

is the output of (5.5), (5.6).

Definition 5.1 *Suppose that $\mathcal{F}$ und $\mathcal{G}$ are quadratic forms on $\mathcal{Y}_1 \times \mathcal{U}$. The* class *of nonlinearities $\mathfrak{N}(\mathcal{F}, \mathcal{G})$ for (5.5), (5.6) defined by $\mathcal{F}(\cdot, \cdot, q)$ and $\mathcal{G}(\cdot, \cdot, q)$ consists of all maps (5.4) such that the following conditions are satisfied:*
For any $T > 0$ and any two functions $\mathbf{y}(\cdot) \in L^2(0, T; Y_1)$ and $\boldsymbol{\xi}(\cdot) \in L^2(0, T; \mathcal{U})$ with

$$\boldsymbol{\xi}(t) \in \boldsymbol{\varphi}(t, \mathcal{C}(q)\mathbf{y}(t), q), \quad a.a.\ t \in [0, T]\,, \tag{5.12}$$

it follows that

$$\mathcal{F}(\mathbf{y}(t), \boldsymbol{\xi}(t), q) \geq 0, \quad a.a.\ t \in [0, T]\,, \tag{5.13}$$

and there exists a continuous function $\Phi : \mathcal{Y}_1 \to \mathbb{R}$ such that

$$\int_s^t \mathcal{G}(\mathbf{y}(\tau), \boldsymbol{\xi}(t), q) d\tau \geq \Phi(\mathcal{C}(q)\mathbf{y}(t)) - \Phi(\mathcal{C}(q)\mathbf{y}(s)) \quad \textit{for all} \quad 0 \leq s < t \leq T. \tag{5.14}$$

In the sequel we need the following assumptions for any $q \in Q$:
(A4) The operator $\mathcal{A}(q) \in \mathcal{L}(\mathcal{Y}_1, \mathcal{Y}_{-1})$ is regular, i.e., for any $T > 0$, $\mathbf{y}_0 \in \mathcal{Y}_1$, $\boldsymbol{\Psi}_T \in \mathcal{Y}_1$ and $\mathbf{f} \in L^2(0, T; \mathcal{Y}_0)$ the solutions of the direct problem

$$\dot{\mathbf{y}} = \mathcal{A}(q)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(0) = \mathbf{y}_0, \quad \text{a.a.} \quad t \in [0, T],$$

and of the dual problem

$$\dot{\boldsymbol{\Psi}} = -\mathcal{A}^*(q)\boldsymbol{\Psi} + \mathbf{f}(t), \quad \boldsymbol{\Psi}(T) = \boldsymbol{\Psi}_T, \quad \text{a.a.} \quad t \in [0, T],$$

are strongly continuous in t in the norm of $\mathcal{Y}_1$.

(A5) The pair $(\mathcal{A}(q), \mathcal{B}(q))$ is L^2-controllable, i.e., for arbitrary $\mathbf{y}_0 \in \mathcal{Y}_0$ there exists a control $\boldsymbol{\xi}(\cdot) \in L^2(0,\infty;\mathcal{U})$ such that the problem

$$\dot{\mathbf{y}} = \mathcal{A}(q)\mathbf{y} + \mathcal{B}(q)\boldsymbol{\xi}, \quad \mathbf{y}(0) = \mathbf{y}_0$$

is well-posed on $[0, +\infty)$.

Definition 5.2 *The variational equation (5.5), (5.6) is said to be absolutely dichotomic in the class $\mathfrak{N}(\mathcal{F},\mathcal{G})$ with respect to the output $\mathbf{r}(\cdot,\boldsymbol{\beta})$ from (5.11) if for any response $\{\mathbf{y}(\cdot),\boldsymbol{\xi}(\cdot)\}$ of (5.5), (5.6) with $\mathbf{y}(0) = \mathbf{y}_0, \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0$ the following is true:*
Either $\mathbf{y}(\cdot)$ is unbounded on $[0,\infty)$ in the $\mathcal{Y}_0$-norm or $\mathbf{y}(\cdot)$ is bounded in $\mathcal{Y}_0$ in this norm and there exist constants c_1 and c_2 (which depend only on $\mathcal{A}(q), \mathcal{B}(q)$ and $\mathfrak{N}(\mathcal{F},\mathcal{G})$) such that

$$\|\mathcal{M}_{\boldsymbol{\beta}}\,\mathbf{y}(\cdot) + \mathcal{E}_{\boldsymbol{\beta}}\,\boldsymbol{\xi}(\cdot)\|^2_{2,\mathcal{R}_{\boldsymbol{\beta}}} \le c_1(\|\mathbf{y}_0\|^2_{\mathcal{Y}_0} + c_2).$$

Theorem 5.1 *Suppose that $\boldsymbol{\varphi} \in \mathfrak{N}(\mathcal{F},\mathcal{G})$ and that for the operators $\mathcal{A}(q)$ and $\mathcal{B}(q)$ the assumptions* **(A4)** *and* **(A5)** *are satisfied. Suppose also that there exists a $\mu > 0$ such that the frequency-domain condition*

$$\mathcal{F}^c(\mathbf{y},\boldsymbol{\xi},q) + \mathcal{G}^c(\mathbf{y},\boldsymbol{\xi},q) - \mu\|\mathcal{M}^c_{\boldsymbol{\beta}}\mathbf{y} + \mathcal{E}^c_{\boldsymbol{\beta}}\,\boldsymbol{\xi}\|^2_{\mathcal{R}_{\boldsymbol{\beta}}} \le 0\,,$$
$$\forall(\mathbf{y},\boldsymbol{\xi}) \in \mathcal{Y}^c_1 \times \mathcal{U}^c : \exists\, \omega \in \mathbb{R} \quad \text{with} \quad i\omega\mathbf{y} = \mathcal{A}^c(q)\mathbf{y} + \mathcal{B}^c(q)\xi$$

is satisfied and the functional

$$J(\mathbf{y}(\cdot),\boldsymbol{\xi}(\cdot),q) = \int_0^\infty [\mathcal{F}^c(\mathbf{y}(\tau),\,\boldsymbol{\xi}(\tau),q) + \mathcal{G}^c(\mathbf{y}(\tau),\boldsymbol{\xi}(\tau),q) - \mu\|\mathcal{M}_{\boldsymbol{\beta}^c}\mathbf{y}(\tau) + \mathcal{E}^c_{\boldsymbol{\beta}}\boldsymbol{\xi}(\tau)\|^2_{\mathcal{R}^c_{\boldsymbol{\beta}}}]d\tau \tag{5.15}$$

is bounded from above on the set

$$\mathfrak{M}_{\mathbf{y}_0} = \{\mathbf{y}(\cdot),\boldsymbol{\xi}(\cdot) : \dot{\mathbf{y}} = \mathcal{A}^c(q)\mathbf{y} + \mathcal{B}^c(q)\boldsymbol{\xi}, \mathbf{y}(0) = \mathbf{y}_0\,, \quad \mathbf{y}(\cdot) \in \mathfrak{W}^c_\infty\,, \quad \boldsymbol{\xi}(\cdot) \in L^2(0,\infty;\mathcal{U}^c)\}$$

for any $\mathbf{y}_0 \in \mathcal{Y}^c_0$. Assume additionally that any potential Φ from the class $\mathfrak{N}(\mathcal{F},\mathcal{G})$ is nonnegative and there exists a constant $c > 0$ such that

$$\Phi(\mathcal{C}(q)\mathbf{y}) \le c\|\mathbf{y}\|^2_{\mathcal{Y}_0}\,, \quad \forall \mathbf{y} \in \mathcal{Y}_0.$$

Then the equation (5.5), (5.6) is absolutely dichotomic with respect to the output $\mathbf{r}(\cdot,\boldsymbol{\beta})$ from (5.11).

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