Frequency domain conditions for the existence of almost periodic solutions in evolutionary variational inequalities

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2. Evolutionary variational inequalities

Suppose that Y_0 is a real Hilbert space with $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ as scalar product resp. norm. Suppose also that $A : \mathcal{D}(A) \subset Y_0$ is a closed (unbounded) densely defined linear operator. The Hilbert space Y_1 is defined as $\mathcal{D}(A)$ equipped with the scalar product

$$(y,\eta)_1 := ((\beta I - A)y,\, (\beta I - A)\eta)_0\,, \quad y,\eta \in \mathcal{D}(A)\,, \tag{1}$$

where $\beta \in \rho(A)$ ($\rho(A)$ is the resolvent set of A) is an arbitrary but fixed number the existence of which we assume.

The Hilbert space Y_{-1} is by definition the completion of Y_0 with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1} z\|_0$. Thus we have the dense and continuous imbedding

$$Y_1 \subset Y_0 \subset Y_{-1} \tag{2}$$

which is called Hilbert space rigging structure. The duality pairing $(\cdot,\cdot)_{-1,1}$ on $Y_1 \times Y_{-1}$ is the unique extension by continuity of the functionals $(\cdot, y)_0$ with $y \in Y_1$ onto Y_{-1}.

If $-\infty \le T_1 < T_2 \le +\infty$ are arbitrary numbers, we define the norm for Bochner measurable functions in $L^2(T_1,T_2;Y_j)\,,\ j = 1, 0, -1$, through

$$\|y\|_{2,j} := \Big(\int_{T_1}^{T_2} \|y(t)\|_j^2 \, dt\Big)^{1/2}\,. \tag{3}$$

For an arbitrary interval J in $\mathbb{R}$ denote by $\mathcal{W}(J)$ the space of functions $y(\cdot) \in L^2_{\mathrm{loc}}(J;Y_1)$ for which $\dot{y}\,(\cdot) \in L^2_{\mathrm{loc}}(J;Y_{-1})$ equipped with the norm defined for any compact interval $[T_1,T_2]$ by

$$\|y\,(\cdot)\|_{\mathcal{W}(T_1,T_2)} := (\|y\,(\cdot)\|_{2,1}^2 + \|\dot{y}\,(\cdot)\|_{2,-1}^2)^{1/2}\,. \tag{4}$$

By an imbedding theorem we can assume that any function from $\mathcal{W}(J)$ belongs to $C(J;Y_0)$. Assume now that Ξ is an other real Hilbert space with scalar product $(\cdot,\cdot)_\Xi$ and norm $\|\cdot\|_\Xi$, respectively, and $J \subset \mathbb{R}$ is an arbitrary interval.

Introduce (with A from above) the linear continuous operators

$$A : Y_1 \to Y_{-1} \quad \text{and} \quad B : \Xi \to Y_{-1} \tag{5}$$

and the maps

$$\varphi : J \times Y_1 \to \Xi\,, \tag{6}$$
$$\psi : Y_1 \to \mathbb{R}_+\,, \tag{7}$$
$$\text{and} \qquad f : J \to Y_{-1}\,. \tag{8}$$

Note that in many applications φ is a material law nonlinearity, B is a control operator, ψ is a contact-type or friction-type functional, and f is a perturbation. Consider for a.a. $t \in J$ the evolutionary variational inequality

$$\begin{aligned}(\dot{y}(t) - Ay(t) - B\varphi(t, y(t)) - f(t)\,,\ \eta - y(t))_{-1,1}& \\ + \psi(\eta) - \psi(y(t)) \geq 0\,, \quad \forall\, \eta \in Y_1\,.&\end{aligned} \tag{9}$$

For any $f \in L^2_{\text{loc}}(J; Y_{-1})$ a function $y(\cdot) \in \mathcal{W}(J) \cap C(J; Y_0)$ is said to be a solution of (9) if this inequality is satisfied for all test functions $\eta \in Y_1$.

In addition, we make the following assumptions.

(A1) For any $t \in J$ the map $\mathcal{A}(t)y := -Ay - B\,\varphi\,(t, y)\ :\ Y_1 \to Y_{-1}$ is semicontinuous, i.e., for any $t \in J$ and any $y, \eta, z \in Y_1$ the $\mathbb{R}$-valued function $\tau \mapsto (\mathcal{A}(t)(y - \tau\eta), z)_{-1,1}$ is continuous.

(A2) For any $\eta \in Y_1$ and any bounded set $S \subset Y_1$ the family of functions $\{(B\varphi\,(\cdot, y), \eta)_{-1,1}\,, y \in S\}$ is equicontinuous on any compact subinterval of J.

(A3) $\varphi\,(\cdot, 0) \equiv 0$ on J and there exist operators $N \in \mathcal{L}(Y_1, \Xi)$ and $M = M^* \in \mathcal{L}(\Xi, \Xi)$ such that

$$\begin{aligned}&(\varphi\,(t, y_1) - \varphi\,(t, y_2), N(y_1, -y_2))_\Xi \\ &\geq\ (\varphi\,(t, y_1) - \varphi\,(t, y_2),\ M\big(\varphi\,(t, y_1) - \varphi\,(t, y_2)\big)_\Xi, \\ &\forall\, t \in J,\ \forall\, y_1, y_2 \in Y_1\,.\end{aligned} \tag{10}$$

(A4) There exists a quadratic form $\mathcal{G}$ on $Y_0 \times \Xi$ and a continuous functional $\Phi : Y_0 \to \mathbb{R}_+$ such that for any $y_1(\cdot), y_2(\cdot) \in L^2_{\text{loc}}(J; Y_0)$ and a.a. $s, t \in J, s < t$, we have

$$\int_s^t \mathcal{G}(y_1(\tau) - y_2(\tau), \varphi(\tau, y_1(\tau)) - \varphi(\tau, y_2(\tau)))d\tau \geq \frac{1}{2}\Phi(y_1(\tau) - y_2(\tau))|_s^t . \tag{11}$$

Furthermore, there are two constants $0 < \rho_1 < \rho_2$ such that

$$\rho_1 \|y\|_0^2 \leq \Phi(y) \leq \rho_2 \|y\|_0^2 , \quad \forall\, y \in Y_0. \tag{12}$$

In addition to **(A1)** – **(A4)** we suppose that there exists a number $\lambda > 0$ such that the following assumptions are satisfied:

(A5) For any $T > 0$ and any $f \in L^2(0, T; Y_{-1})$ the problem $\dot{y} = (A + \lambda I)\, y + f(t), y(0) = y_0$, is well-posed, i.e., for arbitrary $y_0 \in Y_0, f(\cdot) \in L^2(0, T; Y_{-1})$ there exists a unique solution $y(\cdot) \in \mathcal{W}(0, T)$ with $\dot{y}\,(\cdot) \in L^2(0, T; Y_{-1})$ satisfying the equation in a variational sense and depending continuously on the initial data, i.e.,

$$\|y(\cdot)\|^2_{\mathcal{W}(0,T)} \leq c_1 \|y_0\|_0^2 + c_2 \|f(\cdot)\|^2_{2,-1} , \tag{13}$$

where $c_1 > 0$ and $c_2 > 0$ are some constants. Furthermore it is supposed that any solution of $\dot{y} = (A + \lambda I)\, y\, ,\; y\,(0) = y_0$, is exponentially decreasing for $t \to +\infty$, i.e., there exist constants $c_3 > 0$ and $\varepsilon > 0$ such that

$$\|y\,(t)\|_0 \leq c_3\, e^{-\varepsilon t} \|y_0\|_0 \;,\; t > 0 \;. \tag{14}$$

(A6) The operator $A + \lambda I \in \mathcal{L}(Y_1, Y_{-1})$ is regular, i.e., for any $T > 0, y_0 \in Y_1, z_T \in Y_1$ and $f \in L^2(0, T; Y_0)$ the solution of the direct problem

$$\dot{y} = (A + \lambda I)y + f(t), \quad y\,(0) = y_0$$

and of the dual problem

$$\dot{z} = -(A + \lambda I)^* z + f(t), \quad z(0) = z_T$$

are strongly continuous in t in the norm of Y_1.

(A7) The pair $(A + \lambda I, B)$ is L^2-controllable, i.e., for arbitrary $y_0 \in Y_0$ there exists a control $\xi(\cdot) \in L^2(0, +\infty; \Xi)$ such that the problem $\dot{y} = (A + \lambda I)y + B\xi, y(0) = y_0$, is well-posed in the variational sense on $(0, +\infty)$.

(A8) Let denote by H^c and L^c the complexification of a linear space H and a linear operator L, respectively, by
$\chi(s) = (sI^c - A^c)^{-1} B^c$, $s \notin \rho(A^c)$, the transfer operator, and by $\mathcal{G}^c$ the Hermitian extension of $\mathcal{G}$.

There exist a number $\Theta > 0$ such that with ρ_2 from (12) and the imbedding constants γ from $Y_1 \subset Y_0$

$$\begin{aligned} &\Theta\big[\mathrm{Re}\big(\xi, N^c \chi(i\omega - \lambda)\,\xi\big)_{\Xi^c} + (\xi, M^c \xi)_{\Xi^c}\big] \\ &+ \mathcal{G}^c\left(\chi\left(i\omega - \lambda\right)\xi, \xi\right) + \gamma\lambda\rho_2 \|\chi\left(i\omega - \lambda\right)\xi\|^2_{Y_1^c} < 0\,, \\ &\forall\, \omega \in \mathbb{R},\ \forall\, \xi \in \Xi^c\,. \end{aligned} \tag{15}$$

(A9) For any positive $P = P^* \in \mathcal{L}(Y_0, Y_0)$ and $\delta > 0$ which are with γ, ρ_2 and $\Theta > 0$ from **(A8)** solution of the inequality

$$\begin{aligned} &((A + \lambda I)\,y + B\,\xi, Py)_{-1,1} + \Theta\big[(\xi, Ny)_\Xi - (\xi, M\xi)_\Xi\big] \\ &+ \mathcal{G}\,(y, \xi) + \gamma\lambda\,\rho_2\,\|y\|_1^2 \le -\delta\left[\|y\|_1^2 + \|\xi\|_\Xi^2\right] \\ &\forall\, \xi \in \Xi,\ \forall\, y \in Y_1\,, \end{aligned} \tag{16}$$

we have

$$\begin{aligned} &\psi(y_1) - \psi(y_1 - P(y_1 - y_2)) + \psi(y_2) - \psi(y_2 + P(y_1 - y_2)) \ge 0 \\ &\forall\, y_1, y_2 \in Y_1\,, \end{aligned} \tag{17}$$

and on Y_1 the function $\psi_P(y) := \psi(y - Py) - \psi\,(y)$ is convex and lower continuous, i.e., $y_k \to y$ in Y_1 implies
$\psi_P(y) \le \liminf_{k\to\infty} \psi_P(y_k)$.

(A10) For any $y_0 \in Y_0$ the existence of at least one solution $y(\cdot)$ of (9) on $\mathbb{R}_+$ with $y(0) = y_0$ is supposed. The uniqueness to the right and the continuous dependence of solutions on initial states is assumed in the following sense:

a) If y_1, y_2 are two solutions of (9) on $\mathbb{R}_+$ and $y_1(t_0) = y_2(t_0)$ for some $t_0 \geq 0$ then $y_1(t) = y_2(t)$, $\forall\, t \geq t_0$.

b) If $y(\cdot, a_k),\ k = 1, 2, \ldots$, are solutions of (9) with $y(t_0, a_k) = a_k$ on $J_0 = [t_0, t_1]$ or $J_0 = [t_1, t_0]$ and $a_k \to a$ for $k \to \infty$ in Y_0 then there exists a subsequence $k_n \to \infty$ with $y(\cdot, a_{k_n}) \to y$ for $n \to \infty$ in $C(J_0; Y_0)$ and y is a solution of (9) on J_0 with $y(t_0) = a$.

3 Existence of bounded solutions

Let $(E, \|\cdot\|_E)$ be a Banach space. Denote by $C_b(\mathbb{R}; E) \subset C(\mathbb{R}; E)$ the subspace of bounded continuous functions equipped with the norm $\|f\|_{C_b} = \sup\limits_{t \in \mathbb{R}} \|f(t)\|_E$, which gives a Banach space structure.

The space $BS^2(\mathbb{R}; E)$ of *bounded* (with exponent 2) *in the sense of Stepanov functions* is the subspace of all functions f from $L^2_{\text{loc}}(\mathbb{R}; E)$ which have a finite norm

$$\|f\|^2_{S^2} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(\tau)\|^2_E \, d\tau \ .$$

Lemma 3.1 Assume that the assumptions **(A3)** – **(A10)** are satisfied. Then there exists a positive operator $P = P^* \in \mathcal{L}(Y_0, Y_0)$ such that $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and the functional

$$V(y) := \frac{1}{2}(y, Py)_0 + \frac{1}{2}\Phi(y) \ , \quad y \in Y_0 \ ,$$

has the following properties:

a) Suppose that $y(\cdot)$ is an arbitrary solution of (9). Then for any $s, t \in J,\ s \leq t$, we have

$$V\left(y\left(t\right)\right)\big|_s^t + 2\,\lambda \int_s^t\ V\left(y\left(\tau\right)\right)d\tau \leq \int_s^t \left(f(\tau), Py\left(\tau\right)\right)_{-1,1}d\tau\ . \tag{1}$$

b) Suppose that $f \in BS^2(\mathbb{R}_+; Y_{-1})$. Then there exist constants $\alpha > 0$ and $\beta > 0$ such that for any solution $y(\cdot)$ of (9) and any time interval $[s, t] \subset \mathbb{R}_+$ from $\|y(\tau)\|_0 \geq \beta$ on $[s, t]$ it follows that

$$V\left(y\left(\tau\right)\right)\big|_s^t \leq -\alpha \int_s^t \|y(\tau)\|_0^2\ d\tau\,. \tag{2}$$

c) Let $y_1(\cdot), y_2(\cdot)$ be solutions of (9) with $f = f_i \in L^2_{\text{loc}}(J; Y_{-1})$, $i = 1, 2$. Then for any $s, t \in J, s \leq t$, we have

$$\begin{aligned} &V\left(y_1\left(\tau\right) - y_2(\tau)\right)\big|_s^t + 2\lambda \int_s^t V\left(y_1(\tau) - y_2(\tau)\right)\,d\tau \\ &\quad \leq \int_s^t \left(f_1(\tau) - f_2(\tau),\ P\left(y_1(\tau) - y_2(\tau)\right)\right)_{-1,1}d\tau\ . \end{aligned} \tag{3}$$

d) Suppose that $y_1(\cdot),\ y_2(\cdot)$ are two solutions of (9) Then for any $t_0 \in J$ and all $t \geq t_0 (t \leq t_0$, respectively), $t \in J$, we have

$$V\left(y_1(t) - y_2(t)\right) \quad \underset{(\geq)}{\leq} \quad e^{-2\lambda(t-t_0)}\,V\left(y_1(t_0) - y_2(t_0)\right). \tag{4}$$

Proof Due to the assumptions **(A5)** – **(A9)** from the Likhtarnikov-Yakubovich frequency-theorem (Likhtarnikov, Yakubovich; 1976) it follows that there exists an operator $P = P^* \in \mathcal{L}(Y_0, Y_0)$ such that $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and a number $\delta > 0$ such that

$$((A+\lambda I)y + B\,\xi,\ Py)_{-1,1} + \Theta\,[(\xi, Ny)_\Xi - (\xi, M\xi)_\Xi] \\ + \mathcal{G}\,(y,\xi) + \gamma\lambda\rho_2\|y\|_1^2 \le -\delta\left[\,\|y\|_1^2 + \|\,\xi\|_\Xi^2\right] \\ \forall\, y \in Y_1, \quad \forall\, \xi \in \Xi\,. \tag{5}$$

If we put in (5) $\xi = 0$ we get the inequality

$$((A+\lambda I)y,\ Py)_{-1,1} \le -\,\delta\|y\|_1^2\,, \quad \forall\, y \in Y_1\,. \tag{6}$$

Using the assumption **(A5)** it follows from (6) that $P > 0$. Note that P is not necessarily coercive. In order to get this property we consider the functional

$$V\,(y) := \frac{1}{2}\,(y, Py)_0 + \frac{1}{2}\,\Phi(y)\,, \quad \forall\, y \in Y_0\,. \tag{7}$$

Due to the property $P > 0$ and the assumption **(A4)** V is coercive.

Let us prove the assertion a). With the given solution $y(\cdot)$ of (9) we consider for any $t \in J$ the test function $\eta = -Py(t) + y\,(t) \in Y_1$. It follows from (9) that

$$(\dot{y}\,(t), Py\,(t))_{-1,1} + \lambda\,(y\,(t), Py\,(t))_0 \\ -\,((A+\lambda I)y\,(t) + B\,\varphi\,(t, y(t), Py\,(t))_{-1,1} + \psi\,(y(t)) \\ -\,\psi\,(y\,(t) - Py\,(t)) \le (f(t),\ Py\,(t))_{-1,1}\,. \tag{8}$$

Using the estimate (5) we derive from (8) the inequality

$$(\dot{y}\,(t), Py\,(t))_{-1,1} + \lambda\,(y\,(t), Py\,(t))_0 + \Theta\,[(\varphi\,(t, y(t)),\ Ny(t))_\Xi \\ -\,(\varphi(t, y(t)),\ M\varphi\,(t, y(t)))_\Xi] + \mathcal{G}(y(t), \varphi\,(t, y(t))) \\ +\,\gamma\lambda\rho_2\|y(t)\|_1^2 + \delta\,[\|y(t)\|_1^2 + \|\varphi(t, y(t))\|_\Xi^2] \\ +\,\psi\,(y\,(t)) - \psi\,(y(t) - Py\,(t)) \le (f(t), Py\,(t))_{-1,1}\,. \tag{9}$$

Along the solution $y(\cdot)$ we have by **(A3)** and **(A9)**

$$\Theta\,[(\varphi\,(t, y(t)),\ Ny(t))_\Xi - (\varphi(t, y(t)),\ M\varphi\,(t, y(t)))_\Xi] \ge 0\,, \\ \psi\,(y(t)) - \psi\,(y(t) - Py(t)) \ge 0\,, \delta\,[\|y(t)\|_1^2 + \|\varphi(t, y(t))\|_\Xi^2] \ge 0\,. \tag{10}$$

Integrating (9) on a time interval $[s,t],\ s,t \in J$, we get

$$\frac{1}{2}\,(y\,(\tau), Py\,(\tau))_0\,|_s^t + \lambda \int_s^t (y\,(\tau), Py\,(\tau))_0\; d\tau$$
$$+ \int_s^t \mathcal{G}\,(y\,(\tau),\,\varphi\,(\tau, y\,(\tau)))\,d\tau + \gamma\lambda\rho_2 \int_s^t \|y(\tau)\|_1^2\,d\tau$$
$$\le \int_s^t (f\,(\tau), Py\,(\tau))_{-1,1}\,d\tau\;. \qquad (11)$$

From **(A4)** it follows that

$$\int_s^t \mathcal{G}\,(y\,(\tau),\,\varphi\,(\tau, y\,(\tau))) + \gamma\lambda\rho_2 \int_s^t \|y(\tau)\|_1^2\,d\tau$$
$$\ge \frac{1}{2}\,\Phi\,(y\,(\tau))\,|_s^t + \lambda \int_s^t \Phi\,(y\,(\tau))\,d\tau\;. \qquad (12)$$

Taking into account now (11) and (12) we obtain that

$$\left[\frac{1}{2}\,(y\,(\tau), Py\,(\tau))_0 + \frac{1}{2}\,\Phi\,(y\,(\tau))\right]\,|_s^t$$
$$+\,2\,\lambda \int_s^t \left[\frac{1}{2}\,(y\,(\tau), Py\,(\tau))_0 + \frac{1}{2}\,\Phi\,(y\,(\tau))\right]\,d\tau$$
$$\le \int_s^t (f(\tau),\ Py\,(\tau))_{-1,1}\,d\tau\;. \qquad (13)$$

From (13) we conclude that (1) is satisfied.

Now let us prove d). With respect to the solution y_1 we consider the test function $\eta = y_1 + P(y_2 - y_1)$ in order to derive from (9) the inequality (we suppress t in y_i)

$$(\dot{y}_1 - Ay_1 - B\,\varphi\,(t, y_1) - f(t),\ P(y_2 - y_1))_{-1,1}$$
$$+\,\psi\,(y_1\ + P(y_2 - y_1)) - \psi\,(y_1) \ge 0\;. \qquad (14)$$

With respect to the solution y_2 we consider the test function $\eta = y_2 - P\,(y_2 - y_1)$. This gives

$$(\dot{y}_2 - Ay_2 - B\,\varphi\,(t, y_2) - f(t),\ -P(y_2 - y_1))_{-1,1}$$
$$+\,\psi\,(y_2\ - P(y_2 - y_1)) - \psi\,(y_2) \ge 0\;. \qquad (15)$$

If we add the inequalities (14) and (15) we receive

$$
\begin{aligned}
&(\dot y_1 - \dot y_2,\ P(y_2 - y_1))_{-1,1} + (A(y_2 - y_1) + B\,[\varphi\,(t, y_2) - \varphi\,(t, y_1)],\\
&\quad P(y_2 - y_1))_{-1,1} + \psi\,(y_1\ + P(y_2 - y_1)) - \psi\,(y_1)\\
&\quad + \psi\,(y_2 - P(y_2 - y_1)) - \psi\,(y_2) \geq 0 \qquad (16)
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&(\dot y_2 - \dot y_1, P(y_2 - y_1))_{-1,1} - (A(y_2 - y_1) + B\,[\varphi\,(t, y_2) - \varphi\,(t, y_1)],\\
&\quad P(y_2 - y_1))_{-1,1} + \psi\,(y_1) - \psi\,(y_1 + P(y_2 - y_1))\\
&\quad + \psi\,(y_2) - \psi\,(y_2 - P(y_2 - y_1)) \leq 0\ . \qquad (17)
\end{aligned}
$$

From (17) and **(A9)** it follows that

$$
\begin{aligned}
&(\dot y_2 - \dot y_1,\ P(y_2 - y_1))_{-1,1} - (A(y_2 - y_1)\\
&\quad + B\,[\varphi\,(t, y_2) - \varphi\,(t, y_1)]\,,\ P(y_2 - y_1))_{-1,1} \leq 0\ . \qquad (18)
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
&(\dot y_2 - \dot y_1,\ P(y_2 - y_1))_{-1,1} + \lambda(y_2 - y_1,\ P\,(y_2 - y_1))_0\\
&\quad - ((A + \lambda I)\,(y_2 - y_1) + B\,[\varphi\,(t, y_2) - \varphi\,(t, y_1)],\\
&\quad P\,(y_2 - y_1))_{-1,1} \leq 0\ . \qquad (19)
\end{aligned}
$$

We use again use the inequality (5) with $y = y_2 - y_1$ and $\xi = \varphi(t, y_2) - \varphi(t, y_1)$ to derive from (19) the estimate

$$
\begin{aligned}
&(\dot y_2 - \dot y_1,\ P(y_2 - y_1))_{-1,1} + \lambda(y_2 - y_1, P\,(y_2 - y_1))_0\\
&+ \Theta[(\varphi\,(t, y_2) - \varphi(t, y_1),\ N(y_2 - y_1))_\Xi - (\varphi(t, y_2) - \varphi\,(t, y_1),\\
&M(\varphi(t, y_2) - \varphi(t, y_1))_\Xi] + \mathcal{G}\,(y_2 - y_1,\ \varphi\,(t, y_2) - \varphi\,(t, y_1))\\
&+ \gamma\rho_2\lambda\|y_2 - y_1\|_1^2 +\ \delta[\|y_2 - y_1\|_1^2 +\ \|\varphi(t, y_2) - \varphi(t, y_1)\|_\Xi^2] \leq 0\ .\\
&\qquad (20)
\end{aligned}
$$

Along the solution pair y_1, y_2 we have according to **(A3)** the property

$$
\begin{aligned}
&\Theta[(\varphi\,(t, y_2) - \varphi(t, y_1),\ N(y_2 - y_1))_\Xi\\
&\quad - (\varphi(t, y_2) - \varphi\,(t, y_1),\ M(\varphi(t, y_2) - \varphi(t, y_1))_\Xi] \geq 0\ . \ (21)
\end{aligned}
$$

Integration of (20) on $[s,t] \subset J$ under consideration of (21) and $\delta > 0$ gives

$$\frac{1}{2}(y_2 - y_1,\ P(y_2 - y_1))_0|_s^t + \lambda \int_s^t (y_2 - y_1,\ P(y_2 - y_1))_0\, d\tau + \int_s^t \mathcal{G}(y_2 - y_1, \varphi(\tau, y_2) - \varphi(\tau, y_1))\, d\tau + \gamma\rho_2\lambda \int_s^t \|y_2 - y_1\|_1^2\, d\tau \leq 0 \,. \tag{22}$$

From **(A4)** it follows that

$$\int_s^t \mathcal{G}(y_2 - y_1, \varphi(\tau, y_2) - \varphi(\tau, y_1))\, d\tau + \gamma\rho_2\lambda \int_s^t \|y_2 - y_1\|_1^2\, d\tau \geq \frac{1}{2}\Phi(y_2 - y_1)|_s^t + \lambda \int_s^t \Phi(y_2 - y_1)\, d\tau \,. \tag{23}$$

Using (23) we derive from (22) the inequality

$$\frac{1}{2}\big[(y_2 - y_1,\ P(y_2 - y_1))_0 + \Phi(y_2 - y_1)\big]|_s^t + 2\lambda \int_s^t \big[\frac{1}{2}(y_2 - y_1,\ P(y_2 - y_1))_0 + \frac{1}{2}\Phi(y_2 - y_1)\big] d\tau \leq 0 \,. \tag{24}$$

From (24) we conclude that the function

$$m(t) := \frac{1}{2}\big[\big(y_2(t) - y_1(t),\ P(y_2(t) - y_1(t))\big)_0 + \Phi(y_2(t) - y_1(t))\big]$$

satisfies the inequality

$$m(\tau)|_s^t + 2\lambda \int_s^t m(\tau)\, d\tau \leq 0 \,,$$

from which (1) follows immediately.

Lemma 3.2 Suppose that $V : Y_0 \to \mathbb{R}_+$ is a continuous function which satisfies the following properties.

a) There exist constants $0 < \gamma_1 < \gamma_2$ with

$$\gamma_1 \|y\|_0^2 \leq V(y) \leq \gamma_2 \|y\|_0^2 \,, \quad \forall\, y \in Y_0 \,. \tag{25}$$

b) There exist constants $\alpha > 0$ and $\beta > 0$ such that for any solution $y(\cdot)$ of (9) and any time interval $[s,t] \subset \mathbb{R}_+$ from $\|y(\tau)\|_0 \geq \beta$ on $[s,t]$ it follows that

$$V\left(y\left(\tau\right)\right)\big|_s^t \leq -\alpha \int_s^t \|y(\tau)\|_0^2 \, d\tau \, . \tag{26}$$

If $\eta > 0$ is an arbitrary number satisfying the inclusion

$$S := \{y \in Y_0 \; : \; V(y) \leq \eta\} \supset \{y \in Y_0 : \|y\|_0 \leq \beta\} \, , \tag{27}$$

then S is positively invariant for (9) and any solution of (9) enters S in a certain finite time.

Proof a) Suppose that $y(\cdot)$ is a solution of (9) with $y(t_0) \in S$ and $y(t_1) \notin S$ for some $t_1 > t_0$. It follows that $V(y(t_1)) > \eta$ and $\|y\left(t_1\right)\|_0 > \beta$. Denote by t' the maximal time in (t_0, t_1) with $\|y(t')\|_0 = \beta$. On the interval (t', t_1) the inequality $\|y(\tau)\|_0 > \beta$ is satisfied. It follows by (26) that

$$V\left(y\left(\tau\right)\right)\big|_{t'}^{t_1} \leq -\alpha \int_{t'}^{t_1} \|y(\tau)\|_0^2 \, d\tau < 0 \, , \tag{28}$$

and, consequently, $V(y(t_1)) < V(y(t')) \leq \eta$. But this is a contradiction which shows that $y\left(t_1\right) \in S$.

b) Consider a solution $y\left(\cdot\right)$ of (9) with $y(t_0) \notin S$ and $\|y(t_0)\|_0 > \beta$. Assume that $y(t) \notin S$, $\forall \, t \geq t_0$, i.e.,

$$V\left(y\left(t\right)\right) > \eta \quad \text{and} \quad \|y\left(t\right)\|_0 > \beta \, , \quad \forall \, t \geq t_0 \, . \tag{29}$$

From (28) and (29) it follows that for all $t \geq t_0$

$$V\left(y\left(\tau\right)\right)\big|_{t^0}^t \leq -\alpha \int_{t^0}^t \|y(\tau)\|_0^2 \, d\tau \, \leq - \, \alpha \, \beta(t - t_0)$$

and

$$0 < \eta < V\left(y\left(t\right)\right) \leq V(y\left(t_0\right)) - \, \alpha \, \beta(t - t_0) \, .$$

But the last inequality is impossible for large t.

Corollary 3.1 Suppose that the assumptions **(A3)** – **(A10)** are satisfied and

$$f \in BS^2(\mathbb{R}_+; Y_{-1}) \ . \tag{30}$$

Then any solution $y(\cdot)$ of (9) belongs to $C_b\left(\mathbb{R}_+; Y_0\right)$.

Proof From the assumptions **(A3)** – **(A10)** it follows that there exists a continuous function V which satisfies (1). Together with Lemma 3.1 we get the boundedness of any solution in Y_0 on $\mathbb{R}_+$.

(Pankov; 1986, Yakubovich; 1964)

Lemma 3.3 Suppose that there exists a bounded and closed set $S \subset Y_0$ which has the following properties:

a) If for a solution $y(\cdot)$ of (9) we have $y(t_0) \in S$ then $y(t) \in S, \ \forall t \geq t_0$;

b) Any solution $y\left(\cdot\right)$ of (9) enters the set S at a certain time.

Then the inequality (9) has a solution $y \in C_b(\mathbb{R}; Y_0)$ such that $y\left(t\right) \in S, \ \forall\, t \in \mathbb{R}$

Proof Recall that $y(\cdot, a)$ denotes a solution of (9) with $y(0, a) = a$. Put $S_0 := S$ and define for $j = 1, 2, \ldots$ the sets

$$S_j := \{a \in Y_0 : y(-j, a) \in S_0\} \ .$$

It is clear that

$$S_0 \supset S_1 \supset S_2 \supset \cdots \ . \tag{31}$$

Let us show that any set S_j is closed. Suppose for this that $\{a_k\}$ is a sequence of points in S_j with $a_k \to a$ in Y_0. By assumption there exists a subsequence $k_m \to \infty$ and a solution $y(\cdot, a)$ of (9) such that $y(-j, a_{k_m}) \to y(-j, a)$ in Y_0. Since S_0 is closed it follows that $y(-j, a) \in S_0$, i.e., $a \in S_j$.

From (31) and the closedness of S_j it follows that there exists a point $a_0 \in \cap S_j$. For any solution $y\,(\cdot, a_0)$ of (9) we have $y(t, a_0) \in S_0,\ t \geq 0$. From $a_0 \in S_j, j = 1, 2, \dots$, it follows that there exists a solution $y_j(\cdot, a_0)$ with $y_j(-j, a_0) \in S_0,\ y_j(0, a_0) = a_0$, and $y_j(t, a_0) \in S_0,\ \forall t \geq -j$. Choose a subsequence $\{j_m\}$ with $y_{j_m}(-1, a_0) \to a_1$. By assumption we can assume that there exists a solution $y^{(1)}(\cdot)$ of (9) with $j_m(\cdot, a_0) \to y^{(1)}(\cdot)$ on $[-1, 0]$. In addition to this we have $y^{(1)}(0) = a_0$ and $y^{(1)}(-1) = a_1 \in S_0$. Take now a subsequence $\{j_{m_l}\}$ with $y_{j_{m_l}}(-2, a_0) \to a_2 \in S_0$ for $l \to \infty$. Again there is a solution $y^{(2)}(\cdot)$ of (9) such that $y_{j_{m_l}}(\cdot, a_0) \to y^{(2)}(\cdot)$ on $[-2, -1]$, $y^{(2)}(-2) = a_2 \in S_2$, and $y^{(2)}(-1) = a_1$. If we continue this process we get on any interval $[-m, -m+1]$ a solution $y^{(m)}(\cdot)$ satisfying $y^{(m)}(-m) = a_m \in S_0$ and $y^{(m)}(-m+1) = a_{m-1} \in S_0\,, m = 1, 2, \dots$. The bounded on $\mathbb{R}$ solution of (9) is defined by $y(t) = y^{(m)}(t),\ t \in [-m, -m+1]$.

4 Existence of almost periodic solutions

Let $(E, \|\cdot\|_E)$ be a Banach space and let $f : \mathbb{R} \to E$ be continuous. If $\varepsilon > 0$, then a number $T \in \mathbb{R}$ is called ε-*almost period* of f if $\sup_{t\in\mathbb{R}} \|f\,(t+T) - f\,(t)\|_E \leq \varepsilon$. The function f is called *Bohr almost periodic* or *uniformly almost periodic* (shortly $f \in \text{CAP}(\mathbb{R}; E)$ or uniformly a.p.) if for each $\varepsilon > 0$ there is $R > 0$ such that each interval $(r, r+R) \subset \mathbb{R}$ $(r \in \mathbb{R})$ contains at least one ε-almost period of f. For a function $f \in L^2_{\text{loc}}(\mathbb{R}; E)$ define the *Bochner transform* f^b by

$$f^b(t) := f(t+\eta)\ ,\ \eta \in [0,1]\ ,\ t \in \mathbb{R}\ ,$$

as a (continuous) function with values in $L^2(0,1;E)$. A function $f \in BS^2(\mathbb{R}; E)$ is called an *almost periodic function in the sense of Stepanov* (shortly S^2-a.p.) if $f^b \in \text{CAP}(\mathbb{R}; L^2(0,1;E))$. The ε-almost periods of the function f^b are called the ε-almost periods of f. The space of S^2-a.p. functions with values in E is denoted by $S^2(\mathbb{R}; E)$. Obviously, $\text{CAP}(\mathbb{R}; E) \subset S^2(\mathbb{R}; E)$.

In order to derive sufficient conditions for the existence of almost periodic solutions in (9) we need one additional assumption.

(A11) The family of functions $\{\varphi(\cdot, y), y \in Y_1\}$ is uniformly almost periodic on any set $\{y \in Y_1 : \|y\|_1 \leq \text{const}\}$.

Theorem 4.1 Under the assumptions **(A3)** – **(A11)** there exists for any $f \in BS^2(\mathbb{R}; Y_{-1})$ a unique bounded on $\mathbb{R}$ solution $y_*(\cdot)$ of (9). This solution is exponentially stable in the whole, i.e., there exist positive constants $c > 0$ and $\varepsilon > 0$ such that for any other solution y of (9), any $t_0 \in \mathbb{R}$ and any $t \geq t_0$ we have

$$\|y(t) - y_*(t)\|_0 \leq c\, e^{-\varepsilon(t-t_0)} \|y(t_0) - y_*(t_0)\|_0 \,. \tag{1}$$

If φ satisfies **(A11)** and $f \in S^2(\mathbb{R}; Y_{-1})$ then $y_*(\cdot)$ belongs to CAP $(\mathbb{R}; Y_0)$.

Proof (For the case $\varphi(t, y) \equiv \varphi(y)$) Under our assumptions and for $f \in BS^2(\mathbb{R}; Y_{-1})$ the existence of a bounded on $\mathbb{R}$ solution $y_*(\cdot)$ of (9) follows from Lemma 3.3. The exponential stability of $y_*(\cdot)$ results from (4). The inequality (4) implies immediately that $y_*(\cdot)$ is the only bounded on $\mathbb{R}$ solution. Suppose $f \in S^2(\mathbb{R}; Y_{-1})$ and consider an arbitrary ε-almost period of f. Define the function $w(t) := y_*(t + T) - y_*(t)$. Using Lemma 3.1 it is easy to show that there are constants $c_1 > 0$ and $c_2 > 0$ such that for all $t_0 \in \mathbb{R}$ and arbitrary $t \geq t_0$

$$V^{1/2}(w(t)) \leq c_1\, e^{-(t-t_0)}\, V^{1/2}(w(t_0)) + c_2\, \varepsilon \,. \tag{2}$$

If we choose $t_0 \to -\infty$ for any fixed t we get the inequality

$$V^{1/2}(w(t)) \leq c_2\, \varepsilon \,,$$

which shows that T is an $c_2\, \varepsilon$-almost period with respect to the metric $V^{1/2}$.

Example 4.1

$$Y_0 = L^2(0, 1), \quad Y_1 = W^{1,2}(0, 1)$$

$$(u, v)_1 = \int_0^1 (uv + u_x v_x)\, dx \tag{3}$$

$$A : Y_1 \to Y_{-1}\,, (Au,v)_{-1,1} = \int_0^1 (Au)(x)v(x)dx := \\ - \int_0^1 (au_x v_x + buv)\,dx \;, \forall\, u,v \in W^{1,2}(0,1) \tag{4}$$

$$(''\,Au = au - bu_x''\,)$$

$$\Xi = \mathbb{R}\,,\; B : \Xi \to Y_{-1}\,, \\ (B\,\xi,v)_{-1,1} := a\,\xi v(1)\,, \quad \forall\,\xi \in \mathbb{R},\; \forall\, v \in W^{1,2}(0,1) \tag{5}$$

$$(''\,B = a\delta(x-1)''\,)$$

$$u_x(0,t) = 0\,, \quad u_x(1,t) = g(w(t)) + f(t)\,, \tag{6}$$

$g\;:\;\mathbb{R} \to \mathbb{R}$ continuous, $f \in L^2_{\text{loc}}(\mathbb{R}) \cap$ CAP $(\mathbb{R})$

$\psi\;:\;W^{1,2}(0,1) \to \mathbb{R}$

$K\;:\;Y_1 \to \mathbb{R}$ linear continuous, $K(u) = \displaystyle\int_0^1 k(x)u(x,t)\,dx$,

$\varphi\;:\;L^2(0,1) \to \mathbb{R}$ given by

$$u \in L^2(0,1) \mapsto w(\cdot) = K\,(u(\cdot)) \mapsto\; g(w(\cdot)) \in \mathbb{R} \tag{7}$$

$$\exists\,\mu_0 > 0 \quad \forall\, w_1, w_2\;:\; 0 \le (g(w_1) - g(w_2))(w_1 - w_2) \\ \le\; \mu_0 (w_1 - w_2)^2\;, \tag{8}$$

$$\exists\, c_1 > 0 \quad \forall\, w_1, w_2 \in \mathcal{W}(0,T) \quad \forall\, s < t,\; s,t \in (0,T)\;: \\ \int_s^t (\dot{w}_1 - \dot{w}_2)\;(\varphi(w_1) - \varphi(w_2))\,d\tau \ge c_1 |w_1(\tau) - w_2(\tau)|^2\Big|_s^t \tag{9}$$

$$\chi(s) = K(\tilde{u}(x,s))\,,\; s \in \mathbb{C}\,, \\ s\tilde{u} = a\tilde{u}_{xx} - b\tilde{u}\,,\; \tilde{u}_x(0,t) = 0\,,\; \tilde{u}_x(1,t) = 0 \tag{10}$$

$$\chi(s) = K\left(\frac{ab\,\cosh(\frac{1}{a}\sqrt{s+bx})}{\sqrt{s+b}\,\sinh(\frac{1}{a}\sqrt{s+b}}\right) \tag{11}$$

$$\exists\,\Theta > 0 \quad \exists\,\varepsilon > 0 \quad \exists\,\lambda > 0 \quad \forall \omega \in \mathbb{R}\;: \\ \mu_0\,\text{Re}\,\chi(i\omega - \lambda) + \Theta\,\text{Re}\,(i\omega\chi(i\omega - \alpha)) \ge \varepsilon\;, \tag{12}$$

$$\exists\, m > 0 \quad \forall u \in W^{1,2}(0,1)\;\;:\;\; K(u) \ge m\|u\|_1^2 \tag{13}$$

$\Rightarrow$ assumptions of Theorem 4.1 are satisfied