1 Introduction

2 The spinning process as contact problem

2.1 Basic facts from finite-deformation theory

Deformation is a one-parametric family of maps $\{\Phi^t\}_{t\in[0,T]}$

$$\mathbf{\Phi}^t : \Omega \to \mathbb{R}^3 \,.$$

In local coordinates

$$x^i = x^i\left(\overset{\circ}{x}{}^1, \overset{\circ}{x}{}^2, \overset{\circ}{x}{}^3, t\right) \quad , \quad i = 1,2,3 \quad , \quad t \in [0,T]\,.$$

The deformation tensor $\boldsymbol{F}^t$ in the point $\left(\overset{\circ}{x}{}^1, \overset{\circ}{x}{}^2, \overset{\circ}{x}{}^3\right)$ with respect to this basis is defined by

$$\boldsymbol{F}^t = \frac{\partial x^i}{\partial \overset{\circ}{x}{}^j}\, \boldsymbol{e}_i \boldsymbol{e}_j \,.$$

σ^i_j is the first Piola-Kirchhoff tensor. If the columns of $\left(\frac{\partial x^i}{\partial \overset{\circ}{x}{}^j}\right)$ are linearly independent we can write $\sigma^i_j = \sigma^{il}\frac{\partial x^l}{\partial \overset{\circ}{x}{}^j}$. σ^{il} is the second Piola-Kirchhoff tensor.

$$x^i = \overset{\circ}{x}{}^i + u^i(\overset{\circ}{x}{}^1, \overset{\circ}{x}{}^2, \overset{\circ}{x}{}^3, t)$$

Small strain tensor e_{ij}

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial \overset{\circ}{x}{}^j} + \frac{\partial u_j}{\partial \overset{\circ}{x}{}^i}\right)$$

Lagrange strain tensor

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2}\frac{\partial u_k}{\partial \overset{\circ}{x}{}^i}\frac{\partial u_k}{\partial \overset{\circ}{x}{}^j}$$

Equilibrium equation with body forces f_j and material density ρ

$$\frac{\partial \sigma_{\alpha j}}{\partial x^\alpha} + \rho\left(f_j - \frac{\partial^2 u_j}{\partial t^2}\right) = 0$$

The Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial \xi^j} + \frac{\partial g_{jl}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^l}\right) g^{lk}$$

Contravariant differentiation procedure

$$u_{j,i} \equiv \nabla_i u_j = \frac{\partial u_j}{\partial \xi^i} - \Gamma^k_{ij} u_k$$

$$u^j_{\,,i} \equiv \nabla_i u^j = \frac{\partial u^j}{\partial \xi^i} + \Gamma^j_{ki} u^k \,.$$

Small strain and finite strain tensors

$$e_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)$$

$$\varepsilon_{ij} = \frac{1}{2}(g_{ij} - \overset{\circ}{g}_{ij}) = e_{ij} - \frac{1}{2} g^{\alpha\beta}\nabla_i u_\alpha \nabla_j u_\beta \,.$$

2.2 Constitutive law

Plasticity domain K on the blank
$\{\sigma^{ij} : \mathcal{H}(\sigma^{ij}) \leq 0\}$

a) *von Mises material:* $\mathcal{H}(\sigma^{ij}) = \frac{1}{2}s^{ij}s_{ij} - k^2$
$s^{ij} = s_{ij} = \sigma^{ij} - \frac{1}{3}\delta^{ij}\sigma^{kk}$ as deviator of σ^{ij} and $k \neq 0$ is a constant.

b) *Tresca material:* $\mathcal{H}(\sigma^{ij}) = \max|\sigma_i - \sigma_j| - k$
where the maximum is computed over all eigenvalues of the tensor σ^{ij} and $k > 0$ is again a constant.

Total strain is the sum of an elastic part and a plastic part, i.e.

$$\varepsilon_{ij} = \varepsilon^{e}_{ij} + \varepsilon^{p}_{ij}.$$

Generalized Hooke's law

$$\varepsilon^{e}_{ij} = L^{e}_{ijkl}\,\sigma^{kl}$$

Deformation theory

$$\varepsilon^{p}_{ij} = \frac{1}{2\mu}\frac{\sqrt{\frac{1}{2}\,s_{ij}s^{ij}} - k}{\sqrt{\frac{1}{2}\,s_{ij}s^{ij}}}$$

where s_{ij} is the deviator stress and μ is a material constant.
Flow theory

$$\dot{\varepsilon}^{p}_{ij} = \lambda\frac{\partial Y}{\partial \sigma^{ij}} \qquad \dot{\varepsilon}_{ij} = \dot{\varepsilon}^{e}_{ij} + \dot{\varepsilon}^{p}_{ij}.$$

2.3 Plasticity zone in the spinning disc and in the annular disc

Circular disc of thickness h and density ρ rotating with constant angular velocity ω.
(r, φ, z) are the cylindrical coordinates and τ_r and τ_φ are the non-zero Cauchy stress components

$$\frac{\partial \tau_r}{\partial r} + \frac{\tau_r - \tau_\varphi}{r} = -\rho\,\omega^2 r\,.$$

Von Mises-plasticity law possessing the yield condition

$$\tau_r^2 - \tau_r\tau_\varphi + \tau_\varphi^2 = k^2 \qquad (k = \mathrm{const})\,.$$

Connection between Cauchy stresses and small strains e_r and e_φ

$$e_r = \frac{1-2\nu}{E}\tau + \frac{(1+\nu)\Psi}{E}\,(\tau_r - \tau)\,,$$
$$e_\varphi = \frac{1-2\nu}{E}\tau + \frac{(1+\nu)\Psi}{E}\,(\tau_\varphi - \tau)\,.$$

New variables

$$\rho = \frac{r}{r_0} \quad , \quad \tau_\rho = \frac{\tau_r}{k} \quad \text{and} \quad \tau_\varphi := \frac{\tau_\varphi}{k}$$

and the constants

$$a = \frac{r_*}{r_0} \quad , \quad \alpha = \frac{E}{k} \quad \text{and} \quad \lambda = \frac{\rho\,\omega^2 r_0^2}{k}\,.$$

$$\frac{\partial\tau_\rho}{\partial\rho}+\frac{\tau_\rho-\tau_\varphi}{\rho}=-\lambda\rho \quad , \quad 0\le\rho\le 1\,,$$

$$\tau_\rho^2-\tau_\rho\tau\varphi+\tau_\varphi^2=1\,.$$

Ansatz (Arutyunyan et al., 1987)

$$\tau_\rho=\frac{2}{\sqrt{3}}\cos\left(\Phi+\frac{\pi}{6}\right)\,,$$
$$\tau_\varphi=\frac{2}{\sqrt{3}}\cos\left(\Phi+\frac{\pi}{6}\right)$$

ODE problem

$$\rho\,\frac{d\,\Phi}{d\rho}=\frac{\lambda\rho^2\frac{\sqrt{3}}{2}-\sin\Phi}{\sin(\Phi+\frac{\pi}{6})}$$

Compatibility condition

$$\frac{\partial\,e_\rho}{\partial\rho}+\frac{e_\varphi-e_\rho}{\rho}=0\,,$$

Linear ODE problem

$$\frac{1+\nu}{2\alpha\sqrt{3}}\,(\sqrt{3}\sin\omega+\cos\omega)\,\frac{d\Psi}{d\rho}+\Psi\left[\frac{1+\nu}{\alpha\rho}\sin\omega+\right.$$
$$\left.+\;\frac{\sqrt{3}}{2}\frac{d\Psi}{d\rho}\;\left(\sqrt{3}\cos\omega-\sin\omega\right)\right]=\frac{1-2\nu}{\sqrt{3}\alpha}\sin\omega\frac{d\Psi}{d\rho}$$

Elastic domain $a\le\rho\le 1$.
Hooke's law

$$e_\rho=\frac{1}{2}\,(\tau_\rho-\nu\tau_\varphi)\,,$$
$$e_\varphi=\frac{1}{2}\,(\tau_\varphi-\nu\tau_\rho)\,.$$

General solution

$$\tau_\rho=C\Big(1-\frac{1}{\rho^2}\Big)+\frac{\lambda(3+\nu)}{8}\,(1-\rho^2)\,,$$
$$\tau_\rho=C\Big(1+\frac{1}{\rho^2}\Big)+\frac{\lambda}{8}\,[\,3+\nu-(1+3\,\nu)\rho^2]\,.$$

For $\rho=a(\rho)$ we have to guarantee the continuity of the radial and tangential stresses

$$C\Big(1-\tfrac{1}{a^2}\Big)+\tfrac{\lambda(3+\nu)}{8}(1-a^2)\qquad= \\ \tfrac{2}{\sqrt{3}}\cos\left(\Phi(a)+\tfrac{\pi}{6}\right)$$

$$C\Big(1+\tfrac{1}{a^2}\Big)+\tfrac{\lambda}{8}\,[\,3+\nu-(1+3\nu)a^2]\quad= \\ \tfrac{2}{\sqrt{3}}\cos\left(\Phi(a)-\tfrac{\pi}{6}\right)$$

$$\tau_\rho=C\Big(1-\frac{e_\varphi^2}{u_\rho^2}\Big)+\frac{\lambda(3+\nu)}{8}\Big(1-\frac{u_\rho^2}{e_\varphi^2}\Big)\,,$$
$$\tau_\rho=C\Big(1+\frac{e_\varphi^2}{u_\rho^2}\Big)+\frac{\lambda}{8}\left[\,3+\nu-(1+3\nu)\frac{u_\rho^2}{e_\varphi^2}\,\right]\,.$$

Annular plate of thickness h, of outer radius b and of inner radius a, clamped at the inner edge with the outer edge free subjected to uniform radial compression p at the inner edges

For the exactness of the deformation theory it is in the following assumed that (Korovlev, 1971)

$$\frac{a}{b} \geq 0.37$$

Lamé formula for the tangential and radial stresses depending on the actual radius r by (Filin, 1975)

$$\tau_r = \frac{p\,a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

and

$$\tau_\varphi = \frac{p\,a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \ .$$

The stress intensity is

$$\tau_{\text{int}} = \frac{3\beta}{4}\left(\tau_\varphi - \tau_r\right)$$

with $\beta = \frac{2+\sqrt{3}}{2}$.

a) Elastic zone: $\tau_r^e = -\dfrac{2E}{3}A\left(\dfrac{1}{r^2} - \dfrac{1}{b^2}\right)$,

$$\tau_\varphi^e = -\frac{2E}{3}A\left(\frac{1}{r^2} + \frac{1}{b^2}\right) \ ,$$

b) Plastic zone: $\tau_r^p = \dfrac{4k}{3\beta}\left(\ln\dfrac{r}{a} - \kappa\right)$,

$$\tau_\varphi^p = \frac{4k}{3\beta}\left(\ln\frac{r}{a} - \kappa + 1\right)$$

with $\kappa = \frac{3p\beta}{4k}$.
The continuity and compatibility conditions lead to

$$\tau^e_{r_{|r=r_o}} = \tau^p_{r_{|r=r_o}}$$

and

$$\tau_\varphi^e - \tau_r^e = \frac{4k}{3\beta} \, .$$

Critical pressure

$$p_{\text{cr}} = \frac{4k}{3p}\ln\frac{a}{b} \, .$$

$a_s := \frac{b}{a}$ is the spinning ratio in metal forming process.

2.4 Friction theory

Displacement $\boldsymbol{u}$ tangential and normal parts

$$\boldsymbol{u} = \boldsymbol{u}_T + u_N \boldsymbol{n} \ ,$$

where $u_N = \boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u}_T = (\mathrm{id} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{u}$. Surface stress $\boldsymbol{p}$

$$\boldsymbol{p} = \boldsymbol{p}_T + p_N \boldsymbol{n} \ ,$$

p_N is the contact pressure.
The tangential relative velocity $\dot{\boldsymbol{u}}_T$ is decomposed into the adherence part and the slipping part

$$\dot{\boldsymbol{u}}_T = \dot{\boldsymbol{u}}_T^{ad} + \dot{\boldsymbol{u}}_T^{sl} \ .$$

Adherence part

$$\boldsymbol{p}_T = -k \boldsymbol{u}_T^{ad}$$

where k is the elastic contact stiffness.
Slipping part

$$\dot{\boldsymbol{u}}_T^{sl} = -\dot{\gamma} \frac{\partial \Psi}{\partial \boldsymbol{p}_T} \quad ,$$

where the slip potential Ψ determines the direction of slip, and γ is a real function. The state of friction is determined by the slip function Φ and the loading and unloading conditions

loading: $\Phi > 0 \quad , \quad \dot{\gamma} = 0 \, ,$
unloading: $\Phi = 0 \quad , \quad \dot{\gamma} > 0$

and

$\Phi \ \leq \ 0 \Rightarrow$ adherence,
$\Phi \ > \ 0 \Rightarrow$ slipping.

(Ronda and Colville, 1995)

2.5 Large deformation dynamic elastic-plastic contact problem

${}^t\Omega = {}^t\Omega^A \cup {}^t\Omega^B, \ \ {}^t\Gamma = \partial\, {}^t\Omega = \partial\, {}^t\Omega^A \cup \partial\, {}^t\Omega^B = {}^t\Gamma^A \cup {}^t\Gamma^B$

(i) Equilibrium equations

$$\left(\sigma^{kl}\delta^i_l + \sigma^{kl} u^i_{,l}\right)_{,k} + \rho f^i = \rho f^i_{\mathcal{I}} + \rho f^i_C + \rho f^i_Z,$$
$$t \in (0,T), \quad \boldsymbol{x} \in {}^t\Omega \, .$$

(ii) Kinematic boundary conditions (prescribed displacements)

$$u_i(\boldsymbol{x}, t) = U_i(\boldsymbol{x}, t),$$
$$\boldsymbol{x} \in {}^t\Gamma_U = {}^t\Gamma^A_U \cup {}^t\Gamma^B_U, \ t \in (0,T) \, .$$

(iii) Prescribed boundary forces

$$[\sigma^{kl}\delta^i_l + \sigma^{kl} u^i_{,l}] n_k = F^i,$$

(n_1, n_2, n_3 components of $\boldsymbol{n}$)

$$\boldsymbol{x} \in {}^t\Gamma^B_F, \quad t \in (0,T) \, .$$

(iv) Tangential frictional stress

$$\sigma^{ij} n_j - \sigma^{jk} n_j n_k n^i = \mathcal{F}^i,$$
$$\boldsymbol{x} \in {}^t\Gamma_{\mathcal{F}}, \quad t \in (0,T) \, .$$

(v) Initial conditions

$$u_i(\boldsymbol{x}, 0) = U_{0i}(\boldsymbol{x}), \qquad \dot{u}_i(x, 0) = U_{1i}(\boldsymbol{x}) \ .$$

ρ is the density of the material, f^i are body forces, f^i_J are inertia forces, f^i_C are Coriolis forces and f^i_Z denotes the centripetal forces:

$$f^i_C = {}_{(1)}\triangle^{im}\dot{u}_m \ , \ {}_{(1)}\triangle^{im} = 2\epsilon^{inm}\,\omega_n \ ,$$

$$f^i_Z = {}_{(2)}\triangle^i_m u^m \ , \ {}_{(2)}\triangle^i_m = \epsilon^{inl}\epsilon_{mnk}\,\omega_l\omega^k \ , f^i_J = \frac{\partial^2 u^i}{\partial t^2} \ .$$

(Duvant and Lions, 1972)
Main goals of the analysis:
1. Estimation of the contact area between roller tool and material
2. Derivation of upper bounds for stresses at the inner boundary of the flange ring
3. Test of sufficient conditions for the stability of the considered solution using energy methods

3 Plastic buckling and flutter bifurcations in quasi-static problems

General buckling theory

Consider in $\mathbb{R}^3$ the system

$$(\mathrm{A})\begin{cases} \varepsilon_{ij} = K_{ij}[u_k] & \text{in} \quad {}^t\Omega \quad , \\ L_\alpha\,[\sigma^{ij}\,,\,u_i\,,\,f^i] = 0 & \text{in} \quad {}^t\Omega\ ,\ \alpha = 1,2,\dots \ , \\ \tilde{L}_\beta\,[\sigma^{ij}\,,\,u_i\,,\,U_i\,,F^i] = 0 & \text{on} \quad {}^t\Gamma\ ,\ \beta = 1,2,\dots \ , \end{cases}$$

where $\{\varepsilon_{ij}, \sigma^{ij}, u_k\}$ are generalized strains, stresses and displacements.

Loads are dead loads, forces not depending on displacements.
Fix a time t_0 and consider the variational equation

$$(\mathrm{B})\begin{cases} \delta\varepsilon_{ij} = K^0_{ij}\,[\delta u_k] & \text{in} \quad {}^{t_0+\delta t}\,\Omega \\ L^0_\alpha\,[\delta\sigma^{ij}\,,\,\delta u_i\,,\,\delta f^i] = 0 & \text{in} \quad {}^{t_0+\delta t}\,\Omega\ , \\ \tilde{L}^0_\beta\,[\delta\sigma^{ij}\,,\,\delta u_i\,,\,\delta U_i\,,\delta f^i] = 0 & \text{on} \quad {}^{t_0+\delta t}\,\Gamma\ , \\ (\alpha = 1,2,\dots\ , \quad \beta = 1,2,\dots\ .) \end{cases}$$

Plastic wrinkling is associated with non-uniqueness of solution prolongation at $t = t_0$ (Hill, 1958; Hutchinson, 1974).
Suppose $\{\bar{\varepsilon}_{ij}, \bar{\sigma}^{ij}, \bar{u}_k\}$ and $\{\bar{\bar{\varepsilon}}_{ij}, \bar{\bar{\sigma}}^{ij}, \bar{\bar{u}}_k\}$ are two solutions starting at t_0:

$$\begin{aligned} \triangle\,\varepsilon_{ij} &= \bar{\bar{\varepsilon}}_{ij} - \bar{\varepsilon}_{ij} = \delta\varepsilon_{ij}\ , \\ \triangle\,\sigma^{ij} &= \bar{\bar{\sigma}}^{ij} - \bar{\sigma}^{ij} = \delta\sigma^{ij}\ , \\ \triangle u_i &= \bar{\bar{u}}_i - \bar{u}_i \end{aligned}$$

Homogenous perturbational system

$$(\mathrm{C})\begin{cases} \triangle\,\varepsilon_{ij} = K^0_{ij}\,[\triangle u_k] & \text{in} \quad {}^{t_0+\delta t}\,\Omega\ , \\ L^0_\alpha\,[\triangle\sigma^{ij}\,,\,\triangle u_i\,,\,0] = 0 & \text{in} \quad {}^{t_0+\delta t}\,\Omega\ , \\ \tilde{L}^\alpha_\beta\,[\triangle\sigma^{ij}\,,\,\triangle u_i\,,\,0\,,0] = 0 & \text{on} \quad {}^{t_0+\delta t}\Gamma\ , \\ \triangle\sigma^{ij} = \begin{cases} L^{ijmn}_e\cdot\triangle\varepsilon_{mn} & \text{in} \quad {}^{t_0+\delta t}\,\Omega_e \ (\text{elastic}), \\ L^{ijmn}_p\cdot\triangle\varepsilon_{mn} & \text{in} \quad {}^{t_0+\delta t}\,\Omega_p \ (\text{plastic}). \end{cases} \end{cases}$$

(Klyushnikov, 1980)

3.1 Plastic wrinkling of shells

Averaged stresses (over the shell thickness)

$$N^{ij} = \int_{-h/2}^{h/2} \sigma^{ij} \mathrm{d}z \, ,$$

Bending moments

$$M^{ij} = \int_{-h/2}^{h/2} \sigma^{ij} z \mathrm{d}z \quad (i,j \widehat{=} x,y)$$

Shearing forces

$$Q^{i} = \int_{-h/2}^{h/2} \sigma^{3i} \mathrm{d}z \, .$$

Kirchhoff-Love assumption

$$e_{ij} = \varepsilon_{ij} + z\kappa_{ij} \, , \quad z \in \left[-\tfrac{h}{2}, \tfrac{h}{2}\right] \, , \quad i,j = 1,2$$

$\kappa_{ij} = w_{,ij} \, (= \frac{\partial^2 w}{\partial x_i \partial x_j})$
Force equilibrium equations

$$\frac{\partial N^{11}}{\partial x} + \frac{\partial N^{12}}{\partial y} = 0, \quad \frac{\partial N^{22}}{\partial y} + \frac{\partial N^{21}}{\partial x} = 0 \, ,$$

$$\frac{\partial Q^1}{\partial x} + \frac{\partial Q^2}{\partial y} + N^{11}\frac{\partial^2 w}{\partial x^2} + \left(N^{12} + N^{21}\right)\frac{\partial^2 w}{\partial x \partial y} + N^{22}\frac{\partial^2 w}{\partial y^2} = F^3$$

Moment equilibrium equations

$$\begin{aligned} \frac{\partial M^{11}}{\partial x} + \frac{\partial M^{12}}{\partial y} &= Q^1 \, , \\ \frac{\partial M^{22}}{\partial y} + \frac{\partial M^{21}}{\partial x} &= Q^2 \, . \end{aligned}$$

$$N^{ij}_{,j} = 0, \qquad M^{ij}_{,ij} + N^{ij} w_{,ij} = F^3 \, .$$

Denote by $s^{ij} = \mathrm{dev}\,\sigma^{ij} = \sigma^{ij} - \frac{1}{3}\delta^{ij}\sigma^{kk}$ the deviator of the stress tensor, and by $\sigma_{\text{int}} = \sqrt{\frac{1}{2}\, s^{ij} s_{ij}}$ the stress intensity. G is the elastic shear modulus, E is Young's modulus, $E_s = \dfrac{\sigma_{\text{int}}}{\varepsilon_{\text{int}}}$ is the secant modulus, $G' = \dfrac{G}{1 + 3G(\frac{1}{E_s} - \frac{1}{E})}$ is the instantaneous shear modulus and α is the loading index, i.e.,

$$\alpha = \begin{cases} \neq 0 & \text{in} \quad \text{plastic parts} \\ 0 & \text{in} \quad \text{elastic parts.} \end{cases}$$

Material law

$$\delta\sigma^{ij} = 2\,G\, \left[\delta e^{ij} + \delta^{ij}\delta e^{kk} - \tfrac{\alpha}{2\sigma^2_{\text{int}}}\sigma^{ij}\sigma^{mn}\delta e_{mn}\right] \qquad (i,j,k = 1,2)$$

$$
\begin{aligned}
\triangle N^{ij} &= 2G\left[\int_{-h/2}^{h/2}(\triangle e^{ij}+\delta^{ij}\triangle e^{kk})\mathrm{d}z - \int_{\text{plastic part}}\right. \\
\triangle M^{ij} &= 2G\left[\int_{-h/2}^{h/2}(\triangle e^{ij}+\delta^{ij}\triangle e^{kk})z\mathrm{d}z - \int_{\text{plastic part}}\right. \\
\triangle e_{ij} &= \triangle\,\varepsilon_{ij} + z\triangle\kappa_{ij}\,, \\
\triangle N^{ij} &= A^{ijmn}\triangle\,\varepsilon_{mn} \quad \text{and} \quad \triangle M^{ij} = D^{ijmn}\triangle\kappa_{mn},
\end{aligned}
$$

with

$$
\begin{aligned}
A^{ijmn} &= 2Gh\left[\delta^{im}+\delta^{jn}+\delta^{ij}\delta^{mn}-\frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\right], \\
D^{ijmn} &= \frac{Gh^3}{6}\left[\delta^{im}\delta^{jn}+\delta^{ij}\delta^{mn}-\frac{\alpha}{2\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\right].
\end{aligned}
$$

Special case $N^{ij} = h\sigma^{ij}$ (constant stress over the plate)
Bifurcation equation for plastic-elastic buckling

$$
\triangle w_{,ijij} - \frac{1}{4}\left(1-\frac{G'}{G}\right)\frac{1}{\sigma_{\text{int}}^2}\sigma^{ij}\sigma^{mn}\triangle w_{,mnij} + \frac{3\varepsilon^{ij}}{Gh^2}\triangle w_{,ij} = 0\,.
$$

(Klyushnikov, 1980; Korovlev, 1971)

3.2 The plastic buckling behaviour of thin plates under constant pressure

a) **Simply supported rectangular plate**

Conditions

$$N^{22} = N^{12} = 0.$$

Writing $N^{11} = -\tau_{\text{int}}h$ we get the bifurcation equation

$$D_2\triangle^2 w - D_3\frac{\partial^4 w}{\partial x^4} + \tau_{\text{int}}h\frac{\partial^2 w}{\partial x^2} = 0$$

where w denotes the displacements in transversal to the plate direction.
Case 1: All edges are freely supported.

$$w(x,y) = A\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\tau_{\text{int}} = \frac{\pi^2}{h}\left[(D_2-D_3)\frac{m^2}{a^2}+2D_2\frac{n^2}{b^2}+D_2\frac{n^4a^2}{m^2b^4}\right].$$

($n=1$ and an elastic-plastic material with $\beta = \frac{E}{E_s}$) (Korovlev, 1971)

$$e_{\text{int}} = \frac{\pi^2}{a}\frac{h^2}{b^2}\left[\frac{1+3\beta}{4}\left(\frac{b}{a}\right)^2 m^2 + \left(\frac{a}{b}\right)^2\frac{1}{m^2}+2\right] \ (m\in\mathbb{N})\,.$$

Case 2: The edges $x=0$ and $x=a$ are supported and the edges $y=\pm b/2$ are free. Under these assumptions the buckling mode can be considered as

$$w(x) = A\sin\frac{m\pi x}{a}$$

Critical load for wrinkling

$$\tau_{\rm int} = \frac{\pi^2}{36}(1+3\beta)\frac{h^2}{a^2}\,.$$

b) **Circular plate under constant inplane pressure**

$$N^{11} = N^{22} = \tau_{\rm int} h \quad \text{and} \quad N^{12} = 0 \quad \text{as}$$
$$(D_2 - D_3)\triangle^2 w + \tau_{\rm int} h \triangle w = 0\,.$$

$\Phi = \triangle w$

$$(D_2 - D_3)\triangle\Phi + \tau_{\rm int}\, h\Phi = 0$$

Case 1: Axisymmetric plastic buckling

$\Phi = C J_0(r)$ where $J_0(r)$ is the Bessel function of degree 0.

Case 2: Non-axisymmetric plastic buckling

Buckling mode

$$\Phi(r,\varphi) = R(r)\cos\, n\,\varphi$$

Bifurcation equation

$$R^{''} + \frac{R'}{r} + \left(k^2 - \frac{n^2}{r^2}\right) R = 0$$

The solutions are the Bessel functions of the n-th order $R(r) = C\, J_n(r) \quad (C = \text{const})$.

$$\tau_{\rm int} \ge \frac{a^2 k^2}{36}\,(1+3\alpha)\left(\frac{h}{a}\right)^2\,.$$

c) **Annular plate under inplane pressure**

Suppose there is given an annular plate with $\bar{\sigma}$ the effective stress in the flange, h the thickness of the flange, E the plastic buckling modulus, w the actual flange width, K a material constant. If

$$\frac{\bar{\sigma}}{E} \le K\frac{h^2}{w^2}$$

than no flange wrinkling occurs.
(Kobayashi, 1963)

3.3 Plastic buckling and plastic flutter

Homogenous perturbational system in the presence of inertia forces (Bolotin, 1963)

$$\begin{aligned} \boldsymbol{L u} - \frac{\partial^2 \boldsymbol{u}}{\partial t^2} &= 0 \quad \text{in} \quad \Omega \times (0,T)\,, \\ M\boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma_F \times (0,T)\,, \\ N\boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma_{\mathcal{F}} \times (0,T)\,, \end{aligned}$$

Associated static boundary value problem

$$\begin{aligned} \boldsymbol{L u} &= 0 \quad \text{in} \quad \Omega\,, \\ M\boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma_F\,, \\ N\boldsymbol{u} &= 0 \quad \text{on} \quad \Gamma_{\mathcal{F}}\,. \end{aligned}$$

Vibrational solutions in the form

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{U}(\boldsymbol{x})e^{iwt}$$

where $\boldsymbol{U}$ is an unknown function of the phase variables and $\omega \in \mathbb{C}$ is an unknown frequency. Boundary value problems

$$\begin{aligned} L\boldsymbol{U} + \omega^2\boldsymbol{U} &= 0 \quad \text{in} \quad \Omega\,, \\ M\boldsymbol{U} &= 0 \quad \text{on} \quad \Gamma_F\,, \\ N\boldsymbol{U} &= 0 \quad \text{on} \quad \Gamma_{\mathcal{F}}\,. \end{aligned}$$

Condition for the self-adjointness is

$$\int_\Omega \left\{ \left[L\boldsymbol{U}^{(1)} + \omega^2\boldsymbol{U}^{(1)} \right] \boldsymbol{U}^{(2)} - \left[L\boldsymbol{U}^{(2)} + \omega^2\boldsymbol{U}^{(2)} \right] \boldsymbol{U}^{(1)} \right\} = 0\,,$$

(For all perturbations $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}$ satisfying the boundary conditions.)
Suppose now that for a certain value ω the system has a nontrivial solution. If the system is selfadjoint the associated number ω^2 is real. In this case the loss of stability is statically: we have a buckling bifurcation in the variational system.
Consider the PDE problem

$$(D_2 - D_3)\frac{\partial^4 w}{\partial x^4} + \tau_{\text{int}} h \frac{\partial^2 w}{\partial x^2} = 0\,.$$

Including inertia forces we come to the plastic wave equation (ρ denotes the material density)

$$(D_2 - D_3)\frac{\partial^4 w}{\partial x^4} + \tau_{\text{int}} h \frac{\partial^2 w}{\partial x^2} + \rho\frac{\partial^2 w}{\partial t^2} = 0\,.$$

Assume the initial conditions

$$w(0,t) = w(a,t) = 0\;,\; \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(a,t) = 0\;,\; t > 0.$$

We try to find a wave solution in the form

$$w(x,t) = W(x)e^{i\omega t}$$

where $W(x)$ is an unknown function and $\omega \in \mathbb{C}$ is a parameter to be defined.
We receive the ODE problem

$$(D_2 - D_3)W^{(4)} + \tau_{\text{int}} h W^{(2)} + \rho(-\omega^2)W = 0$$

or

$$W^{(4)} + k^2 W^{(2)} - \widehat{\omega}^2 W = 0\,.$$

With the abbreviations

$$k = \sqrt{\frac{\tau_{\text{int}} h}{D_2 - D_3}} \text{ and } \widehat{\omega} = \omega\sqrt{\frac{\rho}{D_2 - D_3}}$$

the critical intensity τ^*_{int} for dynamic plastic wrinkling is

$$\tau^*_{\text{int}} = {k^*}^2\,\frac{(D_2 - D_3)}{h}\,.$$

ODE model for the impact-contact problem
(Kirdeev et al., 1984)
$m\ddot{y}_1 + c(y_1 + \Delta) + \kappa(y_1 - y_2) = 0,$
$m_1\ddot{y}_2 + c_1 y_2 + \kappa(1,5y_2 - y_1 - 0,5y_3 - 0,5y_4) = m_1 e\omega^2 \sin(\omega t + \varphi),$
$m\ddot{y}_3 + c(y_3 - \Delta) + \kappa(y_3 - 0,5y_2) = 0,$
$m\ddot{y}_4 + c(y_4 + \Delta) + \kappa(y_4 - 0,5y_2) = 0.$
Existence results for plastic and frictional contact problems
Duvant & Lions, 1972; Moreau, 1976; Johnson, 1976; Ciort & Rabier, 1980; Nećas & Hlaváćek, 1981; Temam, 1985; Rabier et al., 1986; Monteiro Marcques, 1994;
Basic results for elasto-plastic stability and wrinkling of shells
Hill, 1958; Il'yushin, 1963; Korovlev, 1971; Hutchinson, 1974; Klyushnikov, 1980; Palmov, 1998;
Non-linear shell theory
Mushtari, 1957; Vlasov, 1958; Donnell, 1976;
Elasto-plastic analysis of flange wrinkling in deep drawing process
Energy methods
Geckeler, 1928; Yu & Johnson, 1982; Yossifon & Tirosh, 1984; Yang & Lee, 1992; Cao, 1999
Hill's bifurcation theory
Fatnassi et al., 1984; Naruse, 1986; Améziane-Hassani & Neale, 1990; Wang et al., 1994; Scherzinger & Triantafyllidis, 2000; Chu & Xu, 2001;
Convential sheet metal spinning
Instability and wrinkling
(bifurcation analysis and energy methods)
Siebel & Dröge, 1954; Reichel, 1958; Avitzur & Yang, 1960; Kolpakciaglu, 1961; Kegg, 1961; Kobayashi, 1963; Wells, 1968; Barkaya, 1974; Kirdeev et al., 1984; Korol'kov, 2001;
Roller pass programming
Mogil'nyi, 1972; Hayama et al., 1991; Korol'kov et al., 1999;
Statistic and time-series analysis
Mogil'nyi & Moisseev, 1979; Kiryanov & Mishunin, 1997; Suliman et al., 2000; Malenichev & Val'ter, 2001;
Stability of a spinning disc with a transverse concentrated load
(Coriolis effect, gyroscopic problems, divergence instability, resonance, dynamic buckling, circumferential waves)
Carlin et al., 1975; Iwan & Moeller, 1976; Padovan, 1978; Sprinivasan & Ramamurti, 1980; Nowinski, 1983; Leung & Pinnington, 1987; Chen & Wong, 1994; Chen & Jhu, 1997; Huang & Kuang, 2001;
Dynamic behavior of oscillatiors with clearance and periodically time-varying forces
(structures with gaps and impacting, chaotic resonance)
Peterka, 1974; Panovka, 1977; Choi & Noah, 1992; Lenci et al., 1994; Goldman & Muszyusha, 1994; Kahraman & Blankenship, 1997;