Time series analysis of elasto-plastic bifurcations based on extremely short observation times

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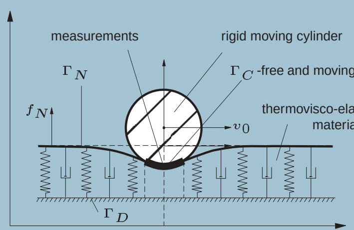
1.Thermovisco-elastoplastic contact

1.1 The mechanical model



1.2 Notation

Suppose $\Omega \subset \mathbb{R}^m$ is a domain (reference configuration of the visco-elastoplastic body), $\Gamma = \partial\Omega$ is the piecewise Lipschitz continuous boundary divided in the three disjunct parts Γ_D (where the body is clamped), Γ_N (where the tractions act) and Γ_C (where the visco-elastoplastic body comes in frictional contact with a rigid moving body).

Assume that $x = (x^1, \dots, x^m)$ is the location in Ω, $t \in \mathbb{R}_+$ is the time, $n = (n^1, \dots, n^m)$ is the unit normal to Γ, $u(x,t) = (u^1(x,t), \dots, u^m(x,t))$ are the displacements, $\Theta = \Theta(x,t)$ is the temperature, $\sigma = (\sigma^{ij})$ is the stress tensor, $f_A = (f_A^1(x,t), \dots, f_A^m(x,t))$ are the body forces in Ω and $\kappa = \kappa(x,t)$ is the density of heat sources.

1.3 Elastoplastic and heat equations

The *equations of motion* and *heat transfer* are given by

$$[\sigma^{kj}(\delta^i_k + u^i_{,k})]_{,j} + f^i_A = \ddot{u}^i \quad \text{in } \Omega \times (0,T), \qquad (1.1)$$

$$\dot{\Theta} - (k^{ij}\Theta_{,j})_{,i} = -c^{ij}\dot{u}_{i,j} + \kappa \quad \text{in } \Omega \times (0,T), \qquad (1.2)$$

where $c^{ij} = c^{ij}(x)$ and $k^{ij} = k^{ij}(x)$ are the tensors of thermal expansion and thermal conductivity, respectively, and the stress tensor is defined by the *thermovisco-elastoplastic stress-strain relation*

$$\sigma^{ij} = a^{ijkl}u_{k,l} + b^{ijkl}\dot{u}_{k,l} - c^{ij}\Theta + \mathcal{P}^{ij}[u_{k,l}, \Theta] \quad \text{in } \Omega \times (0,T), \qquad (1.3)$$

where (a^{ijkl}) and (b^{ijkl}) are the tensors of elastic and viscosity coefficients, respectively, $\{\mathcal{P}^{ij}[\cdot, \Theta]\}_{\Theta > 0}$ is the plastic part given by Θ-dependent hysteresis operators.

As *boundary and initial conditions* we have:

a) *Prescribed displacements and temperature*

$$u = 0 \quad \text{on } \Gamma_D \times (0,T);$$
$$\Theta = \Theta_b \quad \text{on } (\Gamma_D \cup \Gamma_N) \times (0,T); \qquad (1.4)$$

b) *Prescribed boundary forces*

$$\sigma^{ij}n_j = f^i_N \quad \text{on } \Gamma_N, \qquad (1.5)$$

where $f_N = (f^i_N(x,t))$ are the applied tractions;

c) *Frictional stress and temperature on* Γ_C

By Coulomb's law of dry friction

$$|\sigma_\mathcal{T}| \le \mu|\sigma_\mathcal{N}|(1 - \delta|\sigma_\mathcal{N}|)_+ \text{ on } \Gamma_C \times (0,T),$$
$$|\sigma_\mathcal{T}| < \mu|\sigma_\mathcal{N}|(1 - \delta|\sigma_\mathcal{N}|)_+ \Rightarrow \dot{u}_\mathcal{T} = v_0, \qquad (1.6)$$
$$|\sigma_\mathcal{T}| = \mu|\sigma_\mathcal{N}|(1 - \delta|\sigma_\mathcal{N}|)_+ \Rightarrow \dot{u}_\mathcal{T} = v_0 - \lambda\sigma_\mathcal{T},$$

$$k^{ij}\Theta_{,i}n_j = \mu|\sigma_\mathcal{N}|(1 - \delta|\sigma_\mathcal{N}|)_+ s_C(\cdot, |\dot{u}_\mathcal{T} - v_0|) - k_e(\Theta - \Theta_R), \qquad (1.7)$$

where $\sigma_\mathcal{N} = \sigma^{ij}n_in_j$ and $u_\mathcal{N} = u^in_i$ are the normal components of σ and u on Γ, respectively, $\sigma^i_\mathcal{T} = \sigma^{ij}n_j - \sigma_\mathcal{N}n^i$ and $u^i_\mathcal{T} = u^i - u_\mathcal{N}n^i$ are the tangential components of σ and u on Γ, friction coefficient, μ is the friction coefficient, v_0 is the velocity of the moving rigid body, δ is a positive constant, Θ_R is the temperature of the rigid body, $s_C(\cdot, r)$ is a prescribed distance function and k_e is coefficient of heat exchange between elastoplastic body and rigid body. *In general there are no classical solutions for* (1.1)-(1.7).

References

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2. Coupled variational systems

2.1 Scales of Hilbert spaces

A collection of real Hilbert spaces $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ with norm $\|\cdot\|_\alpha$ and scalar product $(\cdot,\cdot)_\alpha$ is called *scale* of Hilbert spaces if the following is true:

(i) For any $\alpha > 0$ the space H_α is continuously embedded into H_β, i.e. $H_\alpha \subset H_\beta$ and there exists a $c_1 > 0$ such that $\|h\|_\beta \le c_1\|h\|_\alpha$, $\forall h \in H_\alpha$, and H_α is dense in H_β ;

(ii) For any $\alpha > 0$ and $h \in H_\alpha$ the linear functional $(\cdot, h)_0$ on H_0 can be continuously extended to a linear continuous functional $(\cdot, h)_{-\alpha,\alpha}$ on $H_{-\alpha}$ satisfying $|(h', h)_{-\alpha,\alpha}| \le \|h'\|_{-\alpha}\|h\|_\alpha$, $\forall h' \in H_{-\alpha}$, $\forall h \in H_\alpha$. Any linear continuous functional l on H_α has the form $l(h) = (h', h)_{-\alpha,\alpha}$ with some $h' \in H_{-\alpha}$, i.e., $H_{-\alpha}$ is isomorphic to the space of linear continuous functionals on H_α.

From (i) it follows that for any $\alpha \in (\beta, \gamma)$ the space H_α is *rigged* by H_β and H_γ, i.e., $H_\gamma \subset H_\alpha \subset H_\beta$ with dense and continuous embeddings. Suppose that $\tilde{H}_1 \subset \tilde{H}_0$ are densely and continuously embedded Hilbert spaces and $a : \tilde{H}_1 \times \tilde{H}_1 \to \mathbb{R}$ is a continuous bilinear form, i.e., there exists a $c_2 > 0$ such that $a(h, h') \le c_2\|h\|_1\|h'\|_1$, $\forall h, h' \in \tilde{H}_1$. Then there exists a scale of Hilbert spaces $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ with $H_1 = \tilde{H}_1$, $H_0 = \tilde{H}_0$ and a linear bounded operator $A : H_1 \to H_{-1}$ such that

$$(Ah, h')_{-1,1} = a(h, h'), \ \forall h, h' \in H_1.$$

Example 2.1 Suppose $\Omega \subset \mathbb{R}^m$ is a domain and N is an arbitrary natural number. $\{H^{(N)}_\alpha\}_{\alpha\in\mathbb{R}}$ is the *scale* of *fractional Sobolev spaces* such that $H^{(N)}_l = W^{l,2}(\Omega)$, $l = 0, 1, \dots, N$, with norms $\|u\|^2_{H^{(N)}_\alpha}$ given by

$$\int_\Omega (|u|^2 + \sum_{|\beta|=1}^{\alpha} |D^\beta u|^2)dx =: \|u\|^2_{W^{\alpha,2}}, \quad \text{if } \alpha \ge 0 \text{ integer},$$

$$\|u\|^2_{W^{k,2}} + \sum_{|\beta|=k}\int_\Omega\int_\Omega \frac{|D^\beta u(x) - D^\beta u(y)|^2}{|x-y|^{k+2\lambda}}dxdy, \quad \text{if } \alpha = k + \lambda > 0,\ k \ge 0 \text{ integer},\ \lambda \in (0,1),$$

$$\sup_{\|v\|_{H^{(N)}_{-\alpha}}=1} \left|\int_\Omega u(x)v(x)dx\right|, \quad \text{if } \alpha < 0.$$

2.2 A simplified contact problem

Suppose $\Omega \subset \mathbb{R}^m$ is a bounded domain, $\partial\Omega$ is smooth, $u = u(x,t)$ and $\Theta = \Theta(x,t)$ are the displacement and the temperature in the elastic body satisfying the system

$$u_{tt} + 2\varepsilon u_t - \Delta u + \alpha u = \xi(t),\ \xi(t) \in \varphi(\Theta(t)), \qquad (2.1)$$

$$\Theta_t - \beta\Delta\Theta + u - \gamma\zeta(t) = 0,\ \zeta(t) = g(\Theta(t)), \qquad (2.2)$$

with $\alpha, \beta, \varepsilon, \gamma$ constants, and the boundary and initial conditions

$$u = 0,\ \Theta = 0 \quad \text{on } \partial\Omega \times (0,T) \qquad (2.3)$$

$$u(\cdot,0) = u_0(\cdot),\ \dot{u}(\cdot,0) = u_1(\cdot),\ \Theta(\cdot,0) = \Theta_0 \text{ in } \Omega. \qquad (2.4)$$

$\varphi : \mathbb{R} \to 2^\mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are nonlinear maps satisfying

$$vg(v) - \xi^2 \ge 0,\ \forall v \in \mathbb{R},\ \forall \xi \in \varphi(v) \qquad (2.5)$$

and $g = G'$, i.e. g has a Fréchet differentiable potential. Suppose $\mathcal{A}$ is the self-adjoint positive-definite operator generated by $(-\Delta)$ with zero boundary conditions and having the domain $\mathcal{D}(\mathcal{A}) = W^{2,2}(\Omega) \cap \mathring{W}^{1,2}(\Omega)$. Introduce the spaces $\mathcal{V}_0 = L^2(\Omega)$, $\mathcal{V}_1 = \mathcal{D}(\mathcal{A}^{1/2})$ and $\mathcal{V}_2 = \mathcal{D}(\mathcal{A})$ with

$$(u,v)_s = (\mathcal{A}^{s/2}u, \mathcal{A}^{s/2}v),\ \forall u, v \in \mathcal{V}_s,\ s = 0,1,2, \qquad (2.6)$$

as scalar product and $Y_s = \mathcal{V}_{s+1} \times \mathcal{V}_s$, $Z_s = \mathcal{V}_{s+1}$, $s = 0, 1$, with the scalar product in Y_s given by

$$((u,v),(\bar{u},\bar{v}))_s = (u,\bar{u})_{s+1} + (v,\bar{v})_s, \quad \forall (u,v),(\bar{u},\bar{v}) \in Y_s. \qquad (2.7)$$

3. Observations for bifurcations

The weak form of (2.1), (2.2) is a *parameter-dependent hybrid system consisting* of a *variational inequality* and a *variational equality* of the type

$$(\dot{y} - A(q)y - B(q)\xi, \eta - y)_{Y_{-1},Y_1} + \Psi(\eta, q) - \Psi(y, q) \ge 0, \qquad (3.1)$$

$$w(t) = C(q)y,\ \xi(t) \in \varphi(t, w(t), v(t), q), \qquad (3.2)$$
$$\forall \eta \in L^2(0,T;Y_1),\ \text{a.e. on } (0,T),$$

$$(\dot{z} - A_1(q)z - B_1(q)\xi, \vartheta)_{Z_{-1},Z_1} = 0, \qquad (3.3)$$

$$v(t) = C_1(q)z,\ \zeta(t) \in g(t, w(t), v(t), q),$$
$$\forall \vartheta \in L^2(0,T;Z_1),\ \text{a.e. on} (0,T). \qquad (3.4)$$

Here $q \in Q$ is a parameter, (Q,d) is a metric space. For any $q \in Q$ we assume that $A(q) \in \mathcal{L}(Y_1, Y_{-1})$, $B(q) \in \mathcal{L}(\Xi, Y_{-1})$, $C(q) \in \mathcal{L}(Y_{-1}, W)$, $\Psi(\cdot, q) : Y_1 \to \mathbb{R}_+$, $\varphi(\cdot,\cdot,\cdot,q) : \mathbb{R}_+ \times W \times \Upsilon \to 2^\Xi$, $A_1(q) \in \mathcal{L}(Z_1, Z_{-1})$, $B_1(q) \in \mathcal{L}(\mathcal{Z}, Z_{-1})$, $g(\cdot,\cdot,\cdot,q) : \mathbb{R}_+ \times W \times \Upsilon \to \mathcal{Z}$, $Y_1, Y_{-1}, Z_1, Z_{-1}, \Xi, W, \mathcal{Z}, \Upsilon$ are real Hilbert spaces. A pair $\{y(\cdot), z(\cdot)\} \in L^2(0,T;Y_1) \times L^2(0,T;Z_1)$ is said to be a *solution* of (3.1)-(3.4) on $(0,T)$ if $\{\dot{y}(\cdot), \dot{z}(\cdot)\} \in L^2(0,T;Y_{-1}) \times L^2(0,T;Z_{-1})$ and there exists a pair $\{\xi(\cdot), \zeta(\cdot)\} \in L^2(0,T;\Xi) \times L^2(0,T;\mathcal{Z})$ such that $\{y(\cdot), z(\cdot), \xi(\cdot), \zeta(\cdot)\}$ satisfies (3.1)-(3.4) for a.e. $t \in (0,T)$ and $\int_0^T \Psi(y(t), q)dt < +\infty$. We assume that for any $T > 0$ such solutions exist.

Definition 3.1 Suppose that $\{S_\alpha\}, \{\tilde{S}_\alpha\}, \{R_\alpha\}$ and $\{\tilde{R}_\alpha\}$ are scales of real Hilbert spaces (*observation* and *output spaces*, respectively) and $D_\alpha \in \mathcal{L}(Y_1, S_\alpha)$, $E_\alpha \in \mathcal{L}(\Xi, S_\alpha)$, $\tilde{D}_\alpha \in \mathcal{L}(Z_1, \tilde{S}_\alpha)$, $\tilde{E}_\alpha \in \mathcal{L}(\mathcal{Z}, \tilde{R}_\alpha)$, $M_\alpha \in \mathcal{L}(Y_1, R_\alpha)$, $N_\alpha \in \mathcal{L}(\Xi, R_\alpha)$, $\tilde{M}_\alpha \in \mathcal{L}(Z_1, \tilde{R}_\alpha)$ and $\tilde{N}_\alpha \in \mathcal{L}(\mathcal{Z}, \tilde{R}_\alpha)$ are scales of linear operators (*observation* and *output operators*, respectively). If $\{y(\cdot), z(\cdot), \xi(\cdot), \zeta(\cdot)\}$ is a response of (3.1)-(3.4) and $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in \mathbb{R}$, are arbitrary scale parameters the function

$$s(\cdot, \alpha, \tilde{\alpha}) = (D_\alpha y(\cdot) + E_\alpha\xi(\cdot), \tilde{D}_{\tilde{\alpha}}z(\cdot) + \tilde{E}_{\tilde{\alpha}}\zeta(\cdot)) \qquad (3.5)$$

is called *observation (measurement* or *time series)* and the function

$$r(\cdot, \beta, \tilde{\beta}) = (M_\beta y(\cdot) + N_p\xi(\cdot), \tilde{M}_{\tilde{\beta}}z(\cdot) + \tilde{N}_{\tilde{\beta}}\zeta(\cdot)), \qquad (3.6)$$

is called (unobservable) output of (3.1)-(3.4). For two responses

$$\{y_i(\cdot), z_i(\cdot), \xi_i(\cdot), \zeta_i(\cdot)\},\ i = 1, 2, \qquad (3.7)$$

of (3.1)-(3.4) and arbitrary scale parameters $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in \mathbb{R}$ we define the deviations

$$\Delta y(\cdot) = y_1(\cdot) - y_2(\cdot),\ \Delta z(\cdot) = z_1(\cdot) - z_2(\cdot),$$
$$\Delta\xi(\cdot) = \xi_1(\cdot) - \xi_2(\cdot),\ \Delta\zeta(\cdot) = \zeta_1(\cdot) - \zeta_2(\cdot), \qquad (3.8)$$

$$\Delta s(\cdot, \alpha)^2 = \|D_\alpha\Delta y(\cdot) + E_\alpha\Delta\xi(\cdot)\|^2_{S_\alpha},$$
$$\Delta\tilde{s}(\cdot, \tilde{\alpha})^2 = \|\tilde{D}_{\tilde{\alpha}}\Delta z(\cdot) + \tilde{E}_{\tilde{\alpha}}\Delta\zeta(\cdot)\|^2_{\tilde{S}_{\tilde{\alpha}}}, \qquad (3.9)$$

$$\Delta r(\cdot, \beta)^2 = \|M_\beta\Delta y(\cdot) + N_\beta\Delta\xi(\cdot)\|^2_{R_\beta},$$
$$\Delta\tilde{r}(\cdot, \tilde{\beta})^2 = \|\tilde{M}_{\tilde{\beta}}\Delta z(\cdot) + \tilde{N}_{\tilde{\beta}}\Delta\zeta(\cdot)\|^2_{\tilde{R}_{\tilde{\beta}}}, \qquad (3.10)$$

Definition 3.2 Suppose that $a > 0$, $b > 0 (a < b)$ and $t_1 > 0$ are numbers. The observation (3.5) is *determining for the bifurcation "loss of* (a, b, t_1)*-stability"* of the output (3.6) at $q = q^*$ if there exist continuous near q^* real-valued functions $\alpha(\cdot)$, $\tilde{\alpha}(\cdot)$, $\beta(\cdot)$ and $\tilde{\beta}(\cdot)$ with the following properties:

a) For $q = q_1$ the observation (3.5) with $\alpha = \alpha(q_1)$, $\tilde{\alpha} = \tilde{\alpha}(q_1)$ is *determining for the* (a, b, t_1)*-stability* of the output (3.6) with $\beta = \beta(q_1)$, $\tilde{\beta} = \tilde{\beta}(q_1)$, i.e., there exists an $\varepsilon_1 = \varepsilon_1(q_1) > 0$ such that for arbitrary two responses (3.7) and their deviations (3.8) - (3.10) which satisfy

$$\Delta r(0, \beta(q_1))^2 + \Delta\tilde{r}(0, \tilde{\beta}(q_1))^2 < a \qquad (3.11)$$

the observation property

$$\int_0^{t_1} [\Delta s(t, \alpha(q_1))^2 + \Delta\tilde{s}(t, \tilde{\alpha}(q_1))^2]dt < \varepsilon_1 \qquad (3.12)$$

implies the output property

$$\Delta r(t, \beta(q_1))^2 + \Delta\tilde{r}(t, \tilde{\beta}(q_1))^2 < b,\ \forall t \in (0, t_1).$$

b) For $q = q_2$ the observation (3.5) with $\alpha = \alpha(q_2)$, $\tilde{\alpha} = \tilde{\alpha}(q_2)$ is *determining for the* (a, b, t_1)*-instability* of the output (3.6) with $\beta = \beta(q_2)$, $\tilde{\beta} = \tilde{\beta}(q_2)$, i.e., there exists an $\varepsilon_2 = \varepsilon_2(q_2) > 0$ such that for arbitrary two responses (3.7) and their deviations (3.8) - (3.10) which satisfy (3.11) the observation property

$$\int_0^{t^*} [\Delta s(t, \alpha(q_2))^2 + \Delta\tilde{s}(t, \tilde{\alpha}(q_2))^2]dt \ge \varepsilon_2$$

for a time $t^* \in (0, t_1)$ implies the output property

$$\Delta r(t^*, \beta(q_2))^2 + \Delta\tilde{r}(t^*, \tilde{\beta}(q_2))^2 \ge b.$$

Remark 3.1 The "loss of (a, b, t_1)-stability"-bifurcation for visco-elastoplastic systems (3.1)-(3.4) means the loss of stability on a finite time interval and is connected with the creation of almost-periodic solutions ([3]). Frequency-domain conditions for observations of this type of bifurcation are derived in [1]. Oberservations that are determining for upper fractal dimension estimates of negatively invariant sets of variational inequalities are considered in [2].

References

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