Dimension-like properties and almost periodicity for cocycles generated by variational inequalities with delay

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1.1 Evolutionary equations in Lur'e form

Consider some *rigged Hilbert space structure* i.e. $\mathbb{H}_1 \subset \mathbb{H}_0 \subset \mathbb{H}_{-1}$ such that embeddings are dense and continuous, $\Xi, \mathbb{W}$ are other Hilbert spaces,

$$A : \mathbb{H}_1 \to \mathbb{H}_{-1}, \quad B : \Xi \to \mathbb{H}_{-1}, \quad C : \mathbb{H}_1 \to \mathbb{W}$$

are bounded linear operators, $g : \mathbb{R} \times \mathbb{W} \to \Xi$ is a nonlinearity. $F \colon \mathbb{R} \to \mathbb{H}_{-1}$ is a perturbation. We consider the Cauchy problem for

$$\begin{aligned} \dot{u} &= Au + Bg(t, Cu(t)) + F(t), \\ u(0) &= u_0 \in \mathbb{H}_0. \end{aligned} \qquad (1)$$

The variational interpretation of above equation means that

$$(\dot{u}(t) - Au(t) - Bg(t, Cu(t)) - F(t), \eta - u(t))_{-1,1} = 0,$$

for all $\eta \in \mathbb{H}_1$, $u(0) = u_0$. (2)

1.2 ODE case

$\mathbb{H}_1 = \mathbb{H}_0 = \mathbb{H}_{-1} = \mathbb{R}^n$, $\mathbb{W} = \mathbb{R}^m$, $\Xi = \mathbb{R}^k$.

1.3 Boundary control system

Consider the equation

$$\begin{aligned} &u_t = au_{xx} - bu,\ 0 < x < 1 \\ &u_x(0,t) = 0,\ u_x(1,t) = g(w(t)),\ u(\cdot,0) = u_0 \\ &g(w(t)) = Cu(x,t) = \int_0^1 c(x)u(x,t)dx,\ c \in L^2(0,1), \end{aligned} \tag{3}$$

where $g : \mathbb{R} \to \mathbb{R}$ and $a > 0, b > 0$ are numbers and define

$\mathbb{H}_0 = L^2(0,1)$, $\mathbb{H}_1 = W^{1,2}(0,1)$, $\mathbb{H}_{-1} = \mathbb{H}_1^*$, $\Xi = \mathbb{R}$,

$A : \mathbb{H}_1 \to \mathbb{H}_{-1}$,

$(Au, v)_{1,-1} = \int_0^1 (Au)(x)v(x)dx = -\int_0^1 (au_x v_x + buv)dx$, for all $u, v \in W^{1,2}(0,1)$,

$B : \Xi \to \mathbb{H}_{-1}$, $B = b\delta(x-1)$.

1.4 Evolutionary variational inequalities

Suppose $\mathbb{E} \subset \mathbb{H} \subset \mathbb{E}^*$ with $A \in \mathcal{L}(\mathbb{E}, \mathbb{E}^*)$. Assume that Ξ and $\mathbb{W}$ are two real Hilbert spaces with the scalar products and norms denoted by $(\cdot,\cdot)_\Xi$, $\|\cdot\|_\Xi$, $(\cdot,\cdot)_\mathbb{W}$, $\|\cdot\|_\mathbb{W}$ respectively. Introduce linear continuous operators

$$B : \Xi \to \mathbb{H}_{-1},\ C : \mathbb{E} \to \mathbb{W}$$

and define a set-valued map g (material law nonlinearity) and a map ϕ (contact-type or friction functional)

$$g : \mathbb{R} \times \mathbb{W} \to 2^{\Xi},\ \phi : \mathbb{E} \to \mathbb{R}_+ \cup \{+\infty\}.$$

Consider the *evolutionary variational inequalitiy*

$$\begin{aligned} &(\dot{u} - Au - B\xi, \eta - u)_{-1,1} + \phi(\eta) - \phi(u) \geq 0,\ \forall \eta \in \mathbb{E} \\ &w(t) = Cu(t),\ \xi(t) \in g(t, w(t)),\ u(0) = u_0 \in \mathbb{H} \end{aligned} \tag{4}$$

1.5 Delayed differential equations.

Suppose that $\mathbb{H}$ is a Hilbert space and denote by $v_{(t)}(\cdot) : [-\tau, 0] \to \mathbb{H}$, $v_{(t)}(s) = v(t+s)$, $s \in [-\tau, 0]$, $A_i : \mathcal{D}(A_i) \subset \mathbb{H} \to \mathbb{H}$, for $0 \le i \le n$.

The equation

$$\dot{v}(t) = \sum_{i=0}^{n} \left[A_i v(t-\tau_i) + Bg(t, Cv_{(t)})\right] + F(t), \; -\tau \le \tau_n < \ldots < \tau_0,$$

$$v(0) = h \in \mathbb{H}, \; v_{(0)} = v_0 \in L^2(-\tau, 0; \mathbb{H}) \tag{5}$$

can be transformed into the equation in $\mathbb{H}_0 = L^2(-\tau, 0; \mathbb{H}) \times \mathbb{H}$ with $(u_1, u_2)_0 = (\{v_1, h_1\}, \{v_2, h_2\})_0 = \int_{-\tau}^{0} (v_1(s), v_2(s))_{\mathbb{H}} ds + (h_1, h_2)_{\mathbb{H}}$ as

$$\begin{aligned} \dot{u}(t) &= \widetilde{A}u(t) + \widetilde{B}\widetilde{g}(t, \widetilde{C}u(t)), \\ u(0) &= u_0 \in \mathbb{H}_0 \end{aligned} \tag{6}$$

2.1 Cocycles in metric spaces

Let $\mathcal{Q}$ be a metric space and let $\vartheta^t\colon \mathcal{Q} \to \mathcal{Q}$ be a dynamical system (flow) on $\mathcal{Q}$.

Let $\mathcal{M}$ be another metric space. A *cocycle* $(\psi, \mathcal{M})$ over $(\vartheta, \mathcal{Q})$ (which in this case is called a *base flow*) is a family of maps $\psi^t(\cdot,\cdot)\colon \mathcal{Q} \times \mathcal{M} \to \mathcal{M}$, where $t \in \mathbb{R}_+$, satysfying [Rokhlin, 1964; Schmalfuss, 1992]

1. $\psi^0(q,\cdot) = \mathrm{id}_{\mathcal{M}}$ for every $q \in \mathcal{Q}$.
2. $\psi^{t+s}(q,u) = \psi^t(\vartheta^s q, \psi^s(q,u))$ for all $q \in \mathcal{Q}, u \in \mathcal{M}$ and $t, s \geq 0$.
3. The map $(t,q,u) \mapsto \psi^t(q,u)$ is continuous.

2. Cocycles in metric spaces

If $\mathcal{Q} = \mathbb{R}$ and ϑ^t is a shift, i. e. $\vartheta^t(s) = s + t$ then the cocycle is called a *process*.

A *complete trajectory* (or *motion*) of the cocycle $(\psi, \mathcal{M})$ over $(\vartheta, \mathcal{Q})$ is a continuous map $u \colon \mathbb{R} \to \mathcal{M}$ such that for some $q \in \mathcal{Q}$ the equality $u(t+s) = \psi^t(\vartheta^s q, u(s))$ holds for all $t \geq 0$ and $s \in \mathbb{R}$. In this case we say that $u(\cdot)$ is *passing through* $u(0)$ *at* q.

Under some reasonable assumptions (on the existence, uniqueness of solutions, etc.) various evolutionary systems generates a cocycle.

2.2 Constructing a cocycle for the microwave heating problem

Consider the problem as a first order system with $\Omega = (0,1)$ and $\varepsilon \equiv \mu \equiv 1$:

$$\begin{aligned}
& w_t = \upsilon - f_t, && (x,t) \in Q_T, \\
& \upsilon_t = \Delta w + \sigma(\theta)\upsilon + f_{tt}, && (x,t) \in Q_T, \\
& \theta_t - \Delta\theta = \sigma(\theta)\left(\upsilon\right)^2, && (x,t) \in Q_T, \\
& w(0,t) = w(1,t) = 0, && 0 < t < T, \\
& \theta(0,t) = \theta(1,t) = 0, && 0 < t < T, \\
& w(x,0) = w_0(x) - f(x,0),\ w_t(x,0) = w_1(x) f(x,0), && x \in \Omega, \\
& \theta(x) = \theta_0(x), && x \in \Omega,
\end{aligned} \tag{7}$$

where $w(x,t) = \int_0^t e(x,\tau)\, d\tau$, $\upsilon = w_t$. Put $u := (w, \upsilon, \theta)$.

2.3 Solution concept

Assumptions:

(A1)
- σ is locally Lipschitz continuous;
- There exist $0 < \sigma_0 \leq \sigma_1$ such that $\sigma_0 \leq \sigma(\theta) \leq \sigma_1$, for $\theta \geq 0$;
- σ is monotonically decreasing;

(A2)
- $w_0 \in H^1(0,1)$, $w_1 \in L^2(0,1)$, $\theta_0 \in W_3^2(0,1)$ and θ_0 is nonnegative a.e. in $(0,1)$
- $f \in C^2(\mathbb{R})$ and there exist $C > 0$: $|f_t| < C$, $|f_{tt}| < C$.

Theorem (Yin)

*Under assumptions **(A1)–(A2)** problem has a unique global weak solution on Q_T for any $T < \infty$. Futhermore $u \in L^\infty(0,T;H^1(0,1))$, $\theta \in W_3^{2,1}((0,1) \times (0,T))$.*

2.2 Constructing a cocycle for the microwave heating problem (continued)

then we can rewrite it as

$$\frac{du}{dt} = Au + Bg(v, \theta) + F(t),$$

$$A = \begin{pmatrix} 0 & I & 0 \\ -\Delta & 0 & 0 \\ 0 & 0 & -\Delta \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 \\ -I & 0 \\ 0 & I \end{pmatrix}, \; \phi(v, \theta) = \begin{pmatrix} \sigma(\theta) v \\ \sigma(\theta) v^2 \end{pmatrix}.$$

Consider the $\vartheta^t : \mathbb{R} \to \mathbb{R}$ as $\vartheta^t(s) = t + s$ and family of mappings $\{\psi^t(t_0, \cdot)\}$ such that

$$\psi^{(\cdot)}(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{M} \to \mathcal{M},$$

$$\psi^t(t_0, u_0) = u(\vartheta^t(t_0), t_0, u_0),$$

for any $t \in \mathbb{R}_+$, $t_0 \in \mathbb{R}_+$, $u_0 \in \mathcal{M}$, where the space $\mathcal{M} = H_0^1(0,1) \times L^2(0,1) \times L^2(0,1)$ with the norm $\|u\|_{\mathcal{M}}^2 = \|(w, v, \theta)\|_{\mathcal{M}}^2 = \|w\|_{H_0^1}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2$.

3.1 Almost periodic functions

Suppose $\mathbb{E}$ is a Banach space. A continuous function $u\colon \mathbb{R} \to \mathbb{E}$ is called *almost periodic* if for every $\varepsilon > 0$ there is $L(\varepsilon) > 0$ such that for every $a \in \mathbb{R}$ we have $[a, a + L(\varepsilon)] \cap \mathcal{T}_\varepsilon(u) \neq \emptyset$. Here $\mathcal{T}_\varepsilon(u)$ denotes the set of ε-almost periods of u, i. e. $\tau \in \mathbb{R}$ such that $\|u(t+\tau) - u(t)\|_\infty \leq \varepsilon$.

Consider the Fourier series of $u(\cdot)$:

$$u(t) \sim \sum_{k=1}^{\infty} U_k e^{i\lambda_k t},$$

where $U_k \in \mathbb{H}$ and $\lambda_k \in \mathbb{R}$. Denote the $\mathbb{Q}$-space generated by the Fourier exponents $\{\lambda_k\}$ by $\mathrm{mod}_{\mathbb{Q}}(u)$.

3.2 Amenable trajectories

Let $\mathbb{H}$ be a Hilbert space. Suppose a cocycle $(\psi, \mathbb{H})$ over $(\vartheta, \mathcal{Q})$ satisfies the following properties

(H1) There is a continuous linear operator $P\colon \mathbb{H} \to \mathbb{H}$, self-adjoint ($P = P^*$) such that $\mathbb{H}$ splits into the direct sum of some P-invariant subspaces $\mathbb{H}^-$ and $\mathbb{H}^+$, i. e. $\mathbb{H} = \mathbb{H}^- \oplus \mathbb{H}^+$, such that $P\big|_{\mathbb{H}^-} < 0$ and $P\big|_{\mathbb{H}^+} > 0$.

(H2) There is an integer $j > 0$ such that $\dim \mathbb{H}^- = j$.

(H3) For $V(u) := (Pu, u)$ and some numbers $\delta > 0$, $\nu > 0$ we have

$$e^{2\nu t_2} V(\psi^{t_2}(q,u) - \psi^{t_2}(q,v)) - e^{2\nu t_1} V(\psi^{t_1}(q,u) - \psi^{t_1}(q,v)) \leq$$
$$\leq -\delta \int_{t_1}^{t_2} e^{2\nu s} |\psi^s(q,u) - \psi^s(q,v)|^2 ds,$$

for every $u, v \in \mathbb{H}$, $q \in \mathcal{Q}$ and $0 \leq t_1 \leq t_2$.

A complete trajectory $u(\cdot)$ of the cocycle is called *amenable* if

$$\int_{-\infty}^{0} e^{2\nu s}|u(s)|^2 ds < \infty.$$

For $q \in \mathcal{Q}$ consider the set $\mathfrak{A}_q$ of $u_0 \in \mathbb{H}$ such that there exists an amenable trajectory of the cocycle passing through u_0 at q.

Consider the orthogonal projector Π onto $\mathbb{H}^-$.

Theorem

The map $\Pi \colon \mathfrak{A}_q \to \Pi\mathfrak{A}_q$ is a homeomorphism.

3.3 Extension of Cartwright's theorem

Suppose $(\mathcal{Q}, \vartheta)$ is a minimal almost periodic flow. Let $\mathrm{mod}_{\mathbb{Q}}(\vartheta)$ be its frequency module.

An almost periodic trajectory $u(\cdot)$ of the cocycle may have frequencies (the Fourier exponents) that do not belong to $\mathrm{mod}_{\mathbb{Q}}(\vartheta)$. We call these frequencies *additional*. Let $\Lambda^C(u;\vartheta)$ be the $\mathbb{Q}$-space generated by the additional frequencies of $u(\cdot)$.

Theorem (Cartwright, 1969; A. M., 2019)

*Suppose that **(H1)-(H3)** hold; then any almost periodic trajectory $u(\cdot)$ satisfies*

$$\dim \Lambda^C(u;\vartheta) \leq j - 1 = \dim \mathbb{H}^- - 1.$$

For example, if $(\mathcal{Q}, \vartheta)$ is a minimal periodic flow and $j = 1$ then any almost periodic trajectory is subharmonic.

3.4 Generalization of Smith's theorem

Consider in addition.

(H4) There exists at least one amenable trajectory at $q \in \mathcal{Q}$.

In applications, a bounded solution of some equation plays the role of an amenable trajectory required in **(H4)**.

(COM) The operator P from **(H1)** is compact.

The following is a generalization of R. A. Smith's theorem on structure of $\mathfrak{A}_q$ (proved by him for ODEs).

Theorem (A. M., 2019)

Suppose that ***(H1)*** *–* ***(H4)*** *and* ***(COM)*** *hold. Then* $\Pi\mathfrak{A}_q = \mathbb{H}^-$ *and, consequently, the map* $\Pi_q = \Pi\big|_{\mathfrak{A}_q} : \mathfrak{A}_q \to \mathbb{H}^-$ *is a homeomorphism.*

3.5 The map Φ

Suppose **(H1)-(H4)** and **(COM)** holds. Define the map $\Phi\colon \mathcal{Q} \times \mathbb{H}^{-} \to \mathbb{H}$, where $\Phi(q, \zeta) := \Pi_q^{-1}(\zeta)$.

(H5) The map Φ is continuous.

If $\mathcal{Q}$ is compact then assumption **(H5)** implies that
(A) Any bounded complete trajectory has precompact range.
(B) The set $\mathfrak{A} = \bigcup_{q \in \mathcal{Q}} \{q\} \times \mathfrak{A}_q$ is closed in $\mathcal{Q} \times \mathbb{H}$.

(H5) holds in the case when $(\mathcal{Q}, \vartheta)$ is a minimal periodic flow. **In the general situation, we do know whether $\mathfrak{A}$ is closed and Φ is continuous**. For a particular class of almost periodic ODEs the continuity of Φ was proved in [A. M., 2019].

3.6 V. V. Zhikov's principle of stationary point for equations in $\mathbb{R}^n$. Suppose we have an almost periodic equation in $\mathbb{R}^n$:

$$\dot{u} = f(t, u)$$

which generates a cocycle over $(\vartheta, \mathcal{H}(f))$. In dimensions $n = 1$ and $n = 2$ V. V. Zhikov's principle of the stationary point [Levitan B. M., Zhikov V. V., 1982] leads to a detailed description of the set of bounded solutions if the **uniform positive Lyapunov stability** holds. Namely,

n=1: all bounded solutions are almost periodic with the frequencies in $\mathrm{mod}_{\mathbb{Q}}(f)$.

n=2: either all bounded solutions are almost periodic (with a common frequency module) or there is only one almost periodic solution and the others bounded solutions are recurrent.

3.7 An extension of V. V. Zhikov's result:

$\dim \mathbb{H}^- = 1$

We suppose that the cocycle is uniformly positively Lyapunov stable, that is

(UPLS) For every $\varepsilon > 0$ and compact $\mathcal{K} \subset \mathbb{H}$ there is a number $\delta > 0$ such that $|\psi^t(q,u) - \psi^t(q,v)|_{\mathbb{H}} \leq \varepsilon$ for every $t \geq 0$ and $q \in \mathcal{Q}$ provided that $u, v \in \mathfrak{A}_q \cap \mathcal{K}$ and $|u - v|_{\mathbb{H}} \leq \delta$

Suppose that $(\mathcal{Q}, \vartheta)$ is a minimal almost periodic flow.

Theorem

Suppose that ***(H1)-(H5)****,* ***(COM)*** *and* ***(UPLS)*** *are satisfied and* $\dim \mathbb{H}^- = 1$*. Then any bounded complete trajectory* $u(\cdot)$ *is almost periodic with* $\mathrm{mod}_{\mathbb{Q}}(u) \subset \mathrm{mod}_{\mathbb{Q}}(\vartheta)$*.*

3.8 An extension of V. V. Zhikov's result:
$\dim \mathbb{H}^- = 2$

Theorem

Suppose that ***(H1)-(H5)****,* ***(COM)*** *and* ***(UPLS)*** *are satisfied and* $\dim \mathbb{H}^- = 2$*. Then there exists an almost periodic trajectory* $u(\cdot)$ *with* $\mathrm{mod}_{\mathbb{Q}}(u) \subset \mathrm{mod}_{\mathbb{Q}}(\vartheta)$*. Moreover, either all bounded complete trajectories are almost periodic with a* ***common enumerable set of frequecies*** *or* $u(\cdot)$ *is the only almost periodic trajectory and the others are recurrent.*

It is interesting whether we can eliminate the second possibility in the above theorem (under additional assumptions).

3.9 Frequency theorem of Yakubovich-Likhtarnikov

Within a rigged Hilbert space structure $\mathbb{H}_1 \subset \mathbb{H}_0 \subset \mathbb{H}_{-1}$ consider the evolutionary problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + Bg(t, Cu(t)) + F(t), \\ u(0) &= u_0 \in \mathbb{H}_0. \end{aligned} \tag{1}$$

How to find an operator $P = P^*$ satisfying **(H1)-(H4)**?

Note that frequency-domain methods are applicable for variational inequalities:

$$(u'(t) + Au(t) - F(t), v - u(t))_{-1,1} + \phi(v) - \phi(u) \geq 0. \tag{4}$$

3. Reduction in evolutionary problems

Suppose that some reasonable conditions are imposed to guarantee the existence and uniqueness of the solutions. And in addition [Yakubovich V. A., Likhtarnikov A. L., 1976]:

(F1) For any $T > 0$ and $f \in L^2(0,T;\mathbb{H}_{-1})$ the problem

$$\dot{u} = (A + \nu I)u + f(t), u(0) = u_0$$

is well-posed.

(F2) The operator $A + \nu I \in \mathcal{L}(\mathbb{H}_1, \mathbb{H}_{-1})$ is regular.

Some kind of monotonicity of the nonlinearity g is assumed as follows. Suppose that there is $M = M^* \in \mathcal{L}(\Xi, \Xi)$ such that

(F3) For all $u_1, u_2 \in \mathbb{H}_1$ we have

$$(g(Cu_1) - g(Cu_2), M(g(Cu_1) - g(Cu_2)))_\Xi \leq (g(Cu_1) - g(Cu_2), C(u_1 - u_2)) \,.$$

We denote by $\mathbb{H}^\mathbb{C}$ and $L^\mathbb{C}$ the complexification of the real Hilber space $\mathbb{H}$ and the real linear operator L respectively.

We consider the *transfer function* $\chi(z) := C^\mathbb{C}(zI^\mathbb{C} - A^\mathbb{C})^{-1}B^\mathbb{C}$, $z \in \rho(A^\mathbb{C})$ of the triple $(A^\mathbb{C}, B^\mathbb{C}, C^\mathbb{C})$.

Theorem

Suppose in addition

1) *The pair* $(A + \nu I, B)$ *is exponentially stabilizable.*
2) *For the equation in* $\mathbb{H}_0$

$$\dot{u} = (A + \nu I)u$$

we have the decomposition $\mathbb{H}_0 = \mathbb{H}^+ \oplus \mathbb{H}^-$ *such that* $\lim_{t \to -\infty} u(t, u_0) = 0$ *for* $u_0 \in \mathbb{H}^-$ *and* $\lim_{t \to +\infty} u(t, u_0) = 0$ *for* $u_0 \in \mathbb{H}^+$.

3) *The frequency-domain condition*

$$\mathrm{Re}(\chi(i\omega - \lambda)\xi, \xi)_{\Xi}^{\mathbb{C}} - (\xi, M^{\mathbb{C}}\xi)_{\Xi}^{\mathbb{C}} < 0$$

is satisfied for all $\omega \in \mathbb{R}$ *with* $\omega \notin \sigma(A^{\mathbb{C}})$ *and all* $\xi \in \Xi^{\mathbb{C}}$, $\xi \neq 0$.

Theorem (continued)

Then there exists a real operator $P = P^* \in \mathcal{L}(\mathbb{H}_{-1};\mathbb{H}_0) \cap \mathcal{L}(\mathbb{H}_0;\mathbb{H}_1)$ *such that* $P\big|_{\mathbb{H}^-} < 0$ *and* $P\big|_{\mathbb{H}^+} > 0$ *and there exists a number* $\delta > 0$ *such that for* $V(u) := (Pu, u)$, $u \in \mathbb{H}_0$ *we have*

$$\frac{d}{dt}V(u_1(t) - u_2(t)) + 2\nu V(u_1(t) - u_2(t)) \leq \\ - 2\delta\|u_1(t) - u_2(t)\|_1^2 \text{ for almost all } t \geq 0,$$

where $u_1(\cdot)$ *and* $u_2(\cdot)$ *is any solutions of* $\dot{u} = Au + Bg(t, Cu) + F(t)$.

If P is given by the above theorem then **(H1)-(H3)** hold. If the inclusion $\mathbb{H}_1 \subset \mathbb{H}_0$ is compact then the operator $P \in \mathcal{L}(\mathbb{H}_0, \mathbb{H}_0)$ is compact, i. e. we have **(H4)**.

4.1 Strongly monotone operators
Suppose $\mathbb{H}$ is a real Hilbert space, $\mathbb{E}$ is a reflexive Banach space with its dual $\mathbb{E}^*$ and the embeddings

$$\mathbb{E} \subset \mathbb{H} \subset \mathbb{E}^* \tag{8}$$

are dense and continuous. By $\langle u, v \rangle$ we denote the pairing between $u \in \mathbb{E}^*$ and $v \in \mathbb{E}$. By $|\cdot|$ we denote the norm in $\mathbb{H}$.
Let the operator (nonlinear, in general) $A\colon \mathbb{E} \to \mathbb{E}^*$ be bounded and *strongly monotone*, i. e. there are constants $M > 0$ and $p > 1$ such that

$$\langle Au - Av, u - v \rangle \geq M|u - v|^p, \text{ for all } u, v \in \mathbb{E}. \tag{9}$$

4.2 Simple evolutionary problem

Let $F\colon \mathbb{R} \to \mathbb{H}_{-1}$ be an $\mathbb{H}_{-1}$-almost periodic function. Consider the evolutionary problem

$$u'(t) + Au(t) = F(t).$$

Usually, such problems give rise to an almost periodic cocycle over the hull $\mathcal{Q} = \mathcal{H}(F) := \mathrm{Cl}\{F(\cdot + s) | s \in \mathbb{R}\}$ with ϑ^t being the shift operator. For every $q \in \mathcal{Q}$ there is a **unique almost periodic trajectory** at q which **attracts** the other trajectories at q. Our purpose is to **estimate the fractal dimension** of this unique almost periodic trajectory closure.

4.3 Generalizations

For evolutionary systems in Lur'e form, i. e.

$$\dot{u} = Au + Bg(t, Cu) + F(t) \qquad (1)$$

it is possible to construct a **new Hilbert space** with the use of **frequency-domain theorem** such that the right-hand of becomes a strongly monotone operator [Kalinin, R. V. 2012].

Note that one may consider a variational inequality

$$(u'(t) + A(t)u(t) - F(t), v - u(t))_{-1,1} + \phi(v) - \phi(u(t)) \geq 0$$

and perturb it with a small term Bu (delayed feedback)

$$(u'(t) + A(t)u(t) + Bu - F(t), v - u(t))_{-1,1} + \phi(v) - \phi(u(t)) \geq 0.$$

For example, $B(u) := \varepsilon u(h(t))$. The dynamics similar to the above discussed still holds [A. A. Pankov1990].

4.4 A general approach

Let $l_u(\varepsilon)$ be the infinum of numbers $L(\varepsilon)$ such that for all $a \in \mathbb{R}$ we have $[a, a + L(\varepsilon)] \cap \mathcal{T}_\varepsilon(u) \neq \emptyset$. The value

$$\mathfrak{Di}(u) := \limsup_{\varepsilon \to 0+} \frac{\ln l_u(\varepsilon)}{-\ln \varepsilon}. \tag{10}$$

is called the *Diophantine dimension* of $u(\cdot)$.

Theorem

Suppose $u(\cdot)$ is α-Hölder, i. e. it satisfies the Hölder condition of order $\alpha \in (0, 1]$; then

$$\overline{\dim}_B \mathcal{M}_u \leq \mathfrak{Di}(u) + \frac{1}{\alpha}. \tag{11}$$

Thus the problem splits into **two subtasks**: **estimating the Diophantine dimension** $\mathfrak{Di}(u)$ and proving some **regularity** of the solution $u(\cdot)$.

4.5 Estimates of $\mathfrak{Di}(u)$

Theorem (K. Naito, 1982)

Suppose there is $u \in C_b(\mathbb{R}; \mathbb{H}_0) \cap L^2_{loc}(\mathbb{R}; \mathbb{H}_1)$ *such that* $u' \in L^2_{loc}(\mathbb{R}; \mathbb{H}_{-1})$ *and*

$$u'(t) + Au(t) = F(t)$$

holds for almost all $t \in \mathbb{R}$*; then* $u(\cdot)$ *is* $\mathbb{H}_0$*-almost periodic and*

$$\mathfrak{Di}(u) \leq (p-1) \cdot \mathfrak{Di}(F).$$

A similar estimate can be provided if one changes A with a periodic operator $A(t)$ or adds a multivalued term $g(t)u$ and considers the corresponding differential inclusion [K. Naito, 1997], or variational inequalities [A. A. Pankov, 1990].

4.6 Regularity

Suppose

1. $\langle Au - Av, u - v\rangle \geq M|u - v|^2$.
2. $\langle Av, v\rangle \geq c_1\|v\|_1^2 + c_2, v \in \mathbb{E}$.
3. $\|Av\|_{-1} \leq c_3\|v\|_1 + c_4, v \in \mathbb{E}$.

These conditions guarantee the existence of a unique $\mathbb{H}$-almost periodic solution $u^*(\cdot)$ which is exponentially stable.

Theorem (A. A. Pankov, 1990)

Suppose $F \in BS_{\infty}^{2,\theta}$; then the unique almost periodic solution belongs to $C_{b,\infty}^{\mu}$ with $\mu = \theta$, i. e. it is θ-Hölder.

As a corollary we have the estimate for the trajectory closure $\mathcal{M}_{u^*} = \mathrm{Cl}(u^*(\mathbb{R}))$:

$$\overline{\dim}_B \mathcal{M}_{u^*} \leq \mathfrak{Di}(F) + \frac{1}{\theta}.$$

4.7 Estimates of $\mathfrak{Di}(F)$

Suppose $F(t) = \Phi(\omega_1 t, \ldots, \omega_m t)$, where $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$ real numbers and $\Phi \colon \mathbb{T}^m \to \mathbb{E}^*$ is α-Hölder.

Suppose for $\omega_1, \ldots, \omega_m$ the Diophantine condition of order $\beta \geq 0$ holds, i. e. there is $C > 0$ such that for all natural b and integer a we have

$$\max_{1 \leq j \leq m} |\omega_j b - a| \geq C b^{-\frac{1+\beta}{m}}. \tag{12}$$

Theorem (A. M., 2019)

In the above assumptions suppose also that $\beta(m-2) < 1$; then

$$\mathfrak{Di}(F) \leq \frac{1}{\alpha} \cdot \frac{(1+\beta)(m-1)}{1-\beta(m-2)}. \tag{13}$$

4.8 The Liouville phenomenon

So, the estimate of $\overline{\dim}_B \mathcal{M}_{u^*}$ looks like

$$\overline{\dim}_B \mathcal{M}_{u^*} \leq \frac{1}{\alpha} \cdot \frac{(1+\beta)(m-1)}{1-\beta(m-2)} + \frac{1}{\theta}. \tag{14}$$

For almost all $\omega_1, \ldots, \omega_m$ we can eliminate the dependence on β since every m-tuples of real numbers satisfy the Diophantine condition of order β for all $\beta > 0$. **For an exclusive set of zero measure the dependence on β still holds** (the Liouville phenomenon). This happens due to the fact that we cannot control the regularity of the representing function of $u^*(\cdot)$, i. e. of some continuous function $\Phi_{u^*} \colon \mathbb{T}^m \to \mathbb{H}$ such that $u^*(t) = \Phi_{u^*}(\omega_1 t, \ldots, \omega_m t)$. It is clear that the fractal dimension of $\mathcal{M}_{u^*} = \Phi_{u^*}(\mathbb{T}^m)$ depends on the regularity of Φ_{u^*}.

4.9 Numerical simulations

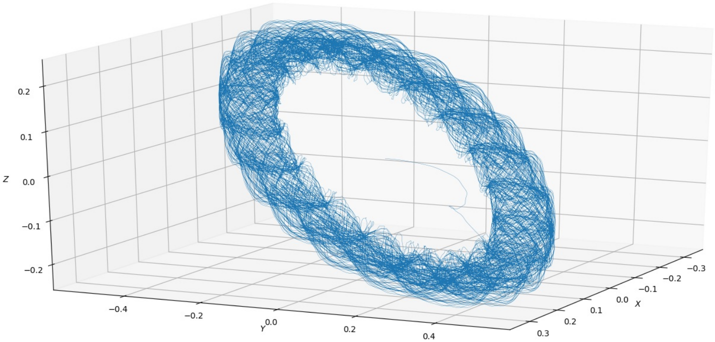

Figure: A forced oscillation (obtained in the perturbed Chua's circuit) with a numerical estimate $\overline{\dim}_B \mathcal{M}_u \approx 2.5$ and upper estimate ≤ 2.5 [M. A., V. R., A. R., 2019]. The upper estimate is given by formula (14) for $\alpha = 2/3$, $m = 2$, $\beta = 0$ and $\theta = 1$.

5. Topological entropy

The method of strongly monotone operators leads to zero topological entropy.

The reduction method allows us to provide an estimate of the topological dimension of fibres of any invariant set (especially, attractors). Namely, for any fibre $\mathcal{A}_q$ of an attractor $\mathcal{A}$ we have

$$\dim_T \mathcal{A}_q \leq j = \dim \mathbb{H}^-.$$

If the map Π_q is bi-Lipschitz (this is the case in finite dimension and can be shown in systems with delay under additional information [R. A. Smith, 1990]) the same estimate holds for the fractal dimension. Thus under suitable conditions we have

$$h_{\text{top}}(\psi, \mathcal{A}) \leq \lambda \cdot j,$$

where λ is a proper Lipschitz-like constant [N. V. Kuznetsov, G. A. Leonov, V. R., 2019].

(1) Cocycles generated by various evolutionary problems, e. g. arising from variational inequalities with delay, are considered.

(2) A global reduction for general cocycles is stated. For almost periodic systems this leads to extension of some results of M. L. Cartwright and V. V. Zhikov.

(3) The application with the use of the Yakubovich-Likhtarnikov frequency theorem for evolutionary systems is considered.

(4) A general approach to study the fractal dimension of almost periodic trajectories is introduced. Its application is shown for cocycles generated by evolutionary problems with a strongly monotone operator.

References

- Cartwright M. L. Almost periodic differential equations and almost periodic flows. *Journal of Differential Equations*, **5**(1), 167–181 (1969).
- Levitan B. M., Zhikov V. V. *Almost Periodic Functions and Differential Equations*. CUP Archive (1982).
- Likhtarnikov A. L., Yakubovich V. A. The frequency theorem for equations of evolutionary type. *Siberian Mathematical Journal*, **17**(5), 790–803 (1976).
- Naito K. Dimension estimate of almost periodic attractors by simultaneous Diophantine approximation. *Journal of Differential Equations*, **141**(1), 179–200 (1997).
- Pankov A. A. Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations. Kluwer Academic Publishers, Dordrecht, (1990).
- Smith R. A. Massera's convergence theorem for periodic nonlinear differential equations. *Journal of Mathematical Analysis and Applications*. **120**(2), 679–708 (1986).

References

- Anikushin, M. M. On the Liouville phenomenon in estimates of fractal dimensions of forced quasi-periodic oscillations, *Vestnik St. Petersb. Univ. Math.*, **52**(3) (2019).
- Anikushin, M. M. On the Smith reduction theorem for almost periodic ODEs satisfying the squeezing property. *Russian Journal of Nonlinear Dynamics*, **15**(1) (2019).
- Anikushin M. M., Reitmann V., Romanov A. Analytical and numerical estimates of fractal dimensions of forced quasi-periodic oscillations in control systems, *Differential Equations and Control Processes* (Differencialnie Uravnenia i Protsesy Upravlenia), **85**(2) (2019).
- Kalinin Y. N., Reitmann V. Almost periodic solutions in control systems with monotone nonlinearities. *Differential Equations and Control Processes* (Differencialnie Uravnenia i Protsesy Upravlenia), **61**(4) (2012).
- Kuznetsov N. V., Leonov G. A., Reitmann V. Attractor Dimension Estimates for Dynamical Systems: Theory and Computation. Switzerland: Springer International Publishing AG (2019).