The global attractor of a multivalued dynamical system generated by a two-phase heating problem

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The microwave heating problem

Consider Maxwell's equations coupled with the heat equation

$$\epsilon(x)\frac{\partial E}{\partial t} + \sigma(x,\theta)E = \mathrm{rot}H, \quad (x,t) \in Q_T = \Omega \times (0,T], \tag{1}$$

$$\mu(x)\frac{\partial H}{\partial t} + \mathrm{rot}E = 0, \quad (x,t) \in Q_T, \tag{2}$$

$$A(\theta)_t - \nabla(k(x,\theta)\nabla\theta) = \sigma(x,\theta)|E|^2, \quad (x,t) \in Q_T, \tag{3}$$

$$\nu \times E(x,t) = 0,\ H \cdot \nu = 0,\ \theta(x,t) = 0, \quad (x,t) \in S_T = \partial\Omega \times (0,T],$$

$$E(x,0) = E_0(x),\ H(x,0) = H_0(x),\ \theta(x,0) = \theta_0(x), \quad x \in \Omega,$$

where Ω is a bounded, open, simply-connected domain of $\mathbb{R}^3$ with a regular boundary $\partial\Omega$ and ν is the outward normal on $\partial\Omega$.
(Yin H.-M., 1998)

Phase change process

Let m be the melting temperature and $A(\theta)$ be the enthalpy operator given by

$$A(\theta) = \begin{cases} \theta - 1, & \text{if } \theta < m, \\ [m-1, m], & \text{if } \theta = m, \\ \theta, & \text{if } \theta > m. \end{cases}$$

Suppose that $\Gamma_T = \{(x,t) \in Q_T \mid \theta(x,t) = m\}$ has Lebesgue measure 0.

Weak solutions

Definition 1

Assume that

$$\iint_{Q_T}(-\epsilon E\cdot\frac{\partial\Upsilon}{\partial t}+\sigma E\cdot\Upsilon)dxdt=\iint_{Q_T}(H\cdot\mathrm{rot}\Upsilon)dxdt+$$

$$+\int_\Omega \epsilon E_0(x)\cdot\Upsilon(x,0)dxdt,$$

$$\iint_{Q_T}(-\mu H\cdot\frac{\partial\Xi}{\partial t}+E\cdot\mathrm{rot}\Xi)dxdt=\int_\Omega(\mu H_0(x)\cdot\Xi(x,0))dx,$$

$$\iint_{Q_T}(-A(\theta)\frac{\partial\eta}{\partial t}+k(x,\theta)\nabla\theta\cdot\nabla\eta)dxdt=\iint_{Q_T}\sigma(\theta)|E|^2\eta dxdt+$$

$$+\int_\Omega A(\theta_0)\eta dx,$$

where $\Upsilon,\Xi\in L^2(0,T;H_0(\mathrm{rot},\Omega))\cap C([0,T],(L^2(\Omega))^3)$,

$$\eta\in H^1(0,T;H^1(\Omega)),$$

$$\Upsilon(x,T)=\Xi(x,T)=0,\ \eta(x,T)=0,\ x\in\Omega.$$

Definition 1 (Cont.)

$(E(x,t), H(x,t), \theta(x,t))$ is called a *weak solution* of the microwave heating problem (1)-(3) if

$$E(\cdot,\cdot), H(\cdot,\cdot) \in C([0,T], (L^2(\Omega))^3),$$
$$\theta(\cdot,\cdot) \in L^2(0,T; H^1(\Omega)) \cap C([0,T]; (L^2(\Omega))^3)$$

and the intergral identities above are true.

previous works:
(Manoranjan V.S., Showalter R., Yin H.-M. 2006) - existence of weak solutions
(Ermakov I., R. V., Skopinov S.N., 2011) - cocycle properties
(Kalinin Yu. N., R. V., 2012) - existence of a.p. solutions
(R. V., Yumaguzin N.Yu., 2012.) - global stability
(R. V., Skopinov S.N., 2014) - stability on a finite time interval

Basic assumptions

Suppose that

(**A1**) $\epsilon(x), \mu(x) \in L^\infty(\Omega)$
$\exists 0 < r_0 < R_0 : 0 < r_0 \le \epsilon(x) \le R_0;\ 0 < r_0 \le \mu(x) \le R_0$ for a.e. $x \in \Omega$,

(**A2**) $\sigma(x,\theta) \in L^\infty(\mathbb{R}_+; L^\infty(\Omega))\ \exists M > 0, \sigma_0 > 0, \sigma_1 > 0 :$
$\sigma_0 \le \sigma(x,\theta) \le \sigma_1,\ \theta\sigma(x,\theta) \le \overline{\sigma}\ \forall(x,\theta) \in \Omega \times [M,\infty)$,

(**A3**) $k_l(\cdot,\cdot), k_s(\cdot,\cdot) \in C^{1+\alpha}(\Omega \times \mathbb{R}_+),\ \alpha \in (0,1],\ \exists r_0' > 0, R_0' > 0 :$
$0 < r_0' \le k_l(x,\theta) \le R_0',\ 0 < r_0' \le k_s(x,\theta) \le R_0',\ \forall(x,\theta) \in \Omega \times [0,\infty)$,

(**A4**) $\theta_0(\cdot) \in L^\infty(\Omega),\ E_0(\cdot), H_0(\cdot) \in (L^2(\Omega))^3$.

Existence of a weak solution

Theorem 1

Under the assumptions ***A1*** *–* ***A4****, the microwave heating problem (1)–(3) has a weak solution on any finite time interval.*

(Manoranjan V.S., Showalter R., Yin H.-M. 2006)

Consider the subspace $\mathcal{D}$ of the solution space:

$$\mathbb{H}_1(\Omega) = H(\mathrm{rot}0, \Omega) \cap H_0(\mathrm{div}0, \Omega),$$
$$\mathcal{D} = \{(E, H, \theta) \in H_0(\mathrm{rot}, \Omega) \times (H(\mathrm{rot}, \Omega) \cap H_0(\mathrm{div}, \Omega)) \times$$
$$\times H_0^1(\Omega); \;\; \mu H \in \mathbb{H}_1(\Omega)^{\perp} \cap H(\mathrm{div}0, \Omega)\},$$

where $\mathbb{H}_1(\Omega)^{\perp}$ is the orthogonal complement of the space $\mathbb{H}_1(\Omega)$ in $L^2(\Omega)^3$.

Theorem 2

Let be initial functions $(E_0, H_0, \theta_0) \in \mathcal{D}$. Then any weak solution $(E(\cdot,\cdot), H(\cdot,\cdot), \theta(\cdot,\cdot))$ of the microwave heating problem (1)–(3) for t $\in (0, T]$ satisfies $(E(\cdot, t), H(\cdot, t), \theta(\cdot, t)) \in \mathcal{D}$.

(Zyrianov. D., R. V., 2017)

Asymptotics of solutions

Theorem 3

For any solution $(E(\cdot,\cdot), H(\cdot,\cdot), \theta(\cdot,\cdot))$ *of the system (1)–(3) with initial functions* $(E_0, H_0, \theta_0) \in \mathcal{D}$ *the following is true:*
1) the interval of existence is $(0, \infty)$*;*
2) $||E(\cdot,t)||^2_{(L^2(\Omega))^3} + ||H(\cdot,t)||^2_{(L^2(\Omega))^3} + ||\theta(\cdot,t)||^2_{L^2(\Omega)} \to 0$ *if* $t \to \infty$*.*

Idea of the proof

1) Orthogonal Hodge decomposition:

$$\mu H = \nabla q + h_1 + \mathrm{rot}\Psi,$$
$$E = -\nabla p - \Psi_t + h_2$$

2) Use of the Lyapunov function ($\lambda, \gamma > 0$ are parameters):

$$\Phi(E,H,\theta) = \frac{1}{2}\int_\Omega (\lambda\epsilon(x)|E(x,t)|^2 + \lambda\mu(x)|H(x,t)|^2 + \gamma A(\theta(x,t))^2)dx.$$

(Zyrianov. D., R. V., 2017)

Let $(\mathcal{M}, \rho)$ be a complete metric space.

Definition 2

Let $\varphi^t : \mathcal{M} \to 2^{\mathcal{M}}, \forall t \in \mathbb{T}$ be a family of multivalued maps. Then $(\{\varphi^t\}_{t \in \mathbb{T}}, (\mathcal{M}, \rho))$ is called *multivalued dynamical system* if the following is true:
1) $\varphi^0(p) = \{p\}$, $\forall p \in \mathcal{M}$,
2) $\varphi^{t_1+t_2}(p) \subset \varphi^{t_1}(\varphi^{t_2}(p))$, $\forall t_1, t_2 \in \mathbb{T}, \forall p \in \mathcal{M}$.

(Melnik V.S., Valero J. 1998)

Multivalued dynamical system for the heating problem

Introduce in the subspace $\mathcal{D}$ the norm

$$||(E, H, \theta)||_{\mathcal{D}} = \max\{||E||_{L^2(\Omega)^3}, ||H||_{L^2(\Omega)^3}, ||\theta||_{L^2(\Omega)}\}.$$

Define the multivalued dynamical system for the microwave heating problem by

$$\varphi : \mathbb{R}_+ \times \mathcal{D} \to 2^{\mathcal{D}},$$

$$\varphi^t(E_0, H_0, \theta_0) = \{(\tilde{E}, \tilde{H}, \tilde{\theta}) \in \mathcal{D} : \exists \text{ solution}(E, H, \theta) \text{ of the system } (1)-(3) \text{ with initial functions } E_0, H_0, \theta_0 \text{ and } E(\cdot, t) = \tilde{E}, H(\cdot, t) = \tilde{H}, \theta(\cdot, t) = \tilde{\theta}\}.$$

$(\{\varphi^t\}_{t\in\mathbb{R}_+}, \mathcal{D})$ satisfies the definition of a multivalued dynamical system (Melnik V.S., Valero J. 1998).

Existence of a global attractor

Theorem 4

Consider the multivalued dynamical system $(\{\varphi^t\}_{t\in\mathbb{R}_+}, \mathcal{D})$ generated by the microwave heating problem. Then this system
1) is continuous,
2) has the compact absorbing set $\mathcal{B}_0$,
3) has the global attractor

$$\mathcal{A} = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \varphi^t(\mathcal{B}_0)},$$

i.e. $\mathcal{A}$ is a closed, bounded, invariant and globally absorbing set for the multivalued dynamical system.

(Zyrianov. D., R. V., 2017)

Consider $\Omega = (0,1)^3$

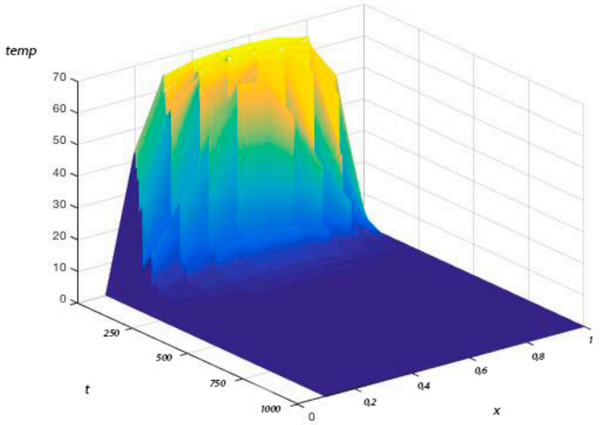

Fig. 1. Temperature at the line $x \in (0,1), y = 0.5, z = 0.5$.

Numerical simulation of the temperature profile

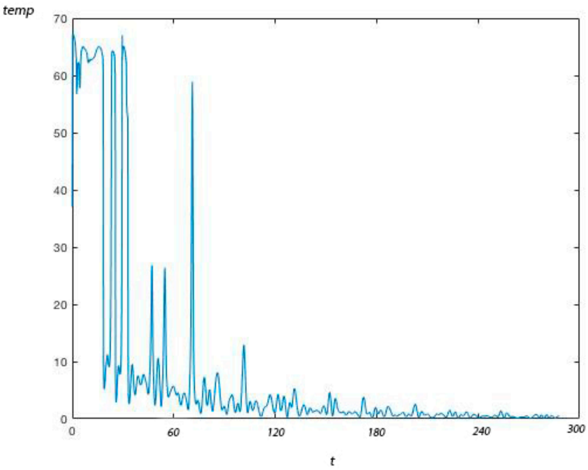

Fig. 2. Temperature at the point $x = 0.5, y = 0.5, z = 0.5$

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