Nonautonomous period-doubling border-collision bifurcations

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Cardiac conduction model

Consider the following system:

$$\begin{cases} A_{k+1} = A_{min} + R_k exp\left(-\frac{A_k+H_k}{\tau_{fat}}\right) + \gamma exp\left(-\frac{H_k}{\tau_{fat}}\right) + \beta_k exp\left(-\frac{H_k}{\tau_{rec}}\right)\Big), \\ R_{k+1} = R_k exp\left(-\frac{A_k+H_k}{\tau_{fat}}\right) + \gamma exp\left(-\frac{H_k}{\tau_{fat}}\right), \end{cases} \quad (1)$$

where:

- $$\beta(A_k) := \beta_k = \begin{cases} 201 - 0.7A_k, \text{ for } A_k < 130, \\ 500 - 3A_k, \text{ for } A_k \geqslant 130; \end{cases}$$
- $A_{min}, \tau_{rec}, \gamma, \tau_{fat}$ are positive constants, $k \in \mathbb{Z}_+$;
- $(A, R) \in \mathbb{R}^2$;
- A_k is the conduction time of the kth impulse;
- H_k is the nodal recovery time during cycle k.
- R_k is the drift in the nodal conduction time of the kth impulse.

Sun J. et al (1995), Maltseva A., R. V. (2014)

Dynamical system generated by an autonomous cardiac conduction system

Consider the following dynamical system:

$$(\{\varphi^k\}_{k\in\mathbb{Z}}, (\mathcal{M}, \rho_{\mathcal{M}})), k \in \mathbb{Z}_+, \tag{2}$$

where

- $\mathcal{M} = \mathbb{R}^2$,
- $\rho_{\mathcal{M}}$ is a standard metric,
- $\varphi^k : \mathcal{M} \to \mathcal{M}, k \in \mathbb{Z}_+$,
- $\varphi(A, R) = \left(A_{min} + R + \beta(A)\exp\left(-\frac{H}{\tau_{rec}}\right), R\exp\left(-\frac{A+H}{\tau_{fat}}\right) + \gamma\exp\left(-\frac{H}{\tau_{fat}}\right)\right)$, $(A, R) \in \mathbb{R}^2$,
- H is a positive constant,
- $\beta(A) := \begin{cases} 201 - 0.7A, \text{ for } A < 130, \\ 500 - 3A, \text{ for } A \geqslant 130. \end{cases}$

Period-doubling border-collision bifurcation

Using the smoothness of the map φ from the left and from the right of the border $\Gamma = \{(130, R) | R \in \mathbb{R}\}$, consider the linearization of φ in these two smooth domains.

Suppose that:

- p is an equilibrium point of dynamical system (2), which exists at the border Γ;
- P and Q are linearization matrices from the left and from the right of the border respectively;
- $\sigma_P = \det P, \sigma_Q = \det Q, \tau_P = \mathrm{tr} P, \tau_Q = \mathrm{tr} Q$.

Theorem 1

Suppose that the equilibrium point p of dynamical system (2) is stable when $H < H_{bif}$ (i.e. $|\sigma_P| < 1, -(1+\sigma_P) < \tau_P < 1+\sigma_P$), and it become unstable when H passing throught H_{bif}. If $|\sigma_P \sigma_Q| < 1, -(1-\sigma_Q)(1-\sigma_P) < \tau_P \tau_Q < (1+\sigma_Q)(1+\sigma_P)$ then a supercritical period-doubling border-collision bifurcation occurs when H is passing throught H_{bif}.

Hassouneh M. A. (2003), Schkolnik D. (2018)

Dissipativity and existence of a global $\mathcal{B}$-attractor

Theorem 2

Dynamical system (2) is dissipative with the dissipativity region:

$$\mathcal{D} = \left[0, \frac{\eta}{1-3\varepsilon}\right] \times \left[0, \frac{\gamma}{1-\lambda}\right], \text{ where}$$

$\eta = A_{min} + \frac{\gamma}{1-\lambda} + 500\exp(-\frac{H_{min}}{\tau_{rec}})$,
$\varepsilon = -\frac{H_{min}}{\tau_{rec}} < \frac{1}{3}, \lambda = \exp(-\frac{A_{min}+H_{min}}{\tau_{fat}}) \neq 1$.

Theorem 3

Dynamical system (2) has a global $\mathcal{B}$-attractor in the form:

$$\mathcal{A}(\mathcal{M}) = \omega(\mathcal{D}) = \cap_{k\in\mathbb{Z}_+}\overline{\cup_{s\geqslant k, s\in\mathbb{Z}_+}\varphi^s(\mathcal{D})}, \quad (3)$$

where $\mathcal{D}$ is the dissipativity region, $\omega(D)$ is the ω-limit set of dynamical system (2).

Maltseva A., V.R. (2016)

Let us consider the following assumptions:

1. in addition to the metric structure $(\mathcal{M}, \rho_M)$ we have the structure of a measurable space $(\mathcal{M}, \mathfrak{B}, \mu)$, where $\mathfrak{B}$ is a σ-algebra over $\mathcal{M}$ and μ is a measure on $\mathfrak{B}$;
2. φ is nonsingular, i.e $\mu(\varphi{-}1(B)) = 0, \ \forall B \in \mathfrak{B} : \mu(B) = 0$.

Definition 1

The Perron-Frobenius operator $P = P_\varphi : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ for the dynamical system (2) is defined by

$$\int_B P\eta d\mu := \int_{\varphi^{-1}(B)} \eta d\mu, \ \forall B \in \mathcal{B}, \forall \eta \in L^1(\mathcal{M}).$$

Perron-Frobenius operator for the dynamical system generated by an autonomous cardiac conduction system

For the case $A < 130$:

$$P\eta(A,R) := \Big|\det J\Big(-\frac{10}{7}\Big((A - A_{min} - R)\exp\Big(\frac{H}{\tau_{rec}}\Big) - 201\Big), (R - \gamma\exp\Big(-\frac{H}{\tau_{fat}}\Big)\exp\Big(\frac{A+H}{\tau_{fat}}\Big)\Big)\Big| \times \eta\Big(-\frac{10}{7}\Big((A - A_{min} - R)\exp\Big(\frac{H}{\tau_{rec}}\Big) - 201\Big), \Big(R - \gamma\exp\Big(-\frac{H}{\tau_{fat}}\Big)\exp\Big(\frac{A+H}{\tau_{fat}}\Big)\Big)\Big),$$

where

$J = \begin{pmatrix} J_{1,1} & J_{1,2} \\ J_{2,1} & J_{2,2} \end{pmatrix}$, $J_{1,1} = -\frac{10}{7}\exp\Big(\frac{H}{\tau_{rec}}\Big)$, $J_{1,2} = \frac{10}{7}\exp\Big(\frac{H}{\tau_{rec}}\Big)$,

$$J_{2,1} = -\frac{10}{7\tau_{fat}}\exp\Big(\frac{H}{\tau_{rec}}\Big)\Big(R - \gamma\exp\Big(-\frac{H}{\tau_{fat}}\Big)\Big)\exp\left(\frac{-\frac{10}{7}\Big((A - A_{min} - R)\exp\Big(\frac{H}{\tau_{rec}}\Big) - 201\Big) + H}{\tau_{fat}}\right),$$

$$J_{2,2} = \Big(1 + \frac{10}{7\tau_{fat}}\exp\Big(\frac{H}{\tau_{rec}}\Big)\Big(R - \gamma\exp\Big(-\frac{H}{\tau_{fat}}\Big)\Big)\Big)\exp\left(\frac{-\frac{10}{7}\Big((A - A_{min} - R)\exp\Big(\frac{H}{\tau_{rec}}\Big) - 201) + H}{\tau_{fat}}\right).$$

Linnik P. L. (2018)

Perron-Frobenius operator for the dynamical system generated by an autonomous cardiac conduction system

For the case $A \geqslant 130$:

$$P\eta(A,R) := \Big|\det J\Big(-\frac{1}{3}\Big((A - A_{min} - R)\exp\Big(\frac{H}{\tau_{rec}}\Big) - 500\Big),$$
$$(R - \gamma\exp\Big(-\frac{H}{\tau_{fat}}\Big)\exp\Big(\frac{A+H}{\tau_{fat}}\Big)\Big)\Big| \times$$
$$\eta\Big(-\frac{1}{3}\Big((A - A_{min} - R)\exp\Big(\frac{H}{\tau_{rec}}\Big) - 500\Big),$$
$$\Big(R - \gamma\exp\Big(-\frac{H}{\tau_{fat}}\Big)\exp\Big(\frac{A+H}{\tau_{fat}}\Big)\Big),$$

where

$$J = \begin{pmatrix} J_{1,1} & J_{1,2} \\ J_{2,1} & J_{2,2} \end{pmatrix}, J_{1,1} = -\tfrac{1}{3}\exp\Big(\tfrac{H}{\tau_{rec}}\Big), J_{1,2} = \tfrac{1}{3}\exp\Big(\tfrac{H}{\tau_{rec}}\Big),$$

$$J_{2,1} = -\tfrac{1}{3\tau_{fat}}\exp\Big(\tfrac{H}{\tau_{rec}}\Big)\Big(R - \gamma\exp\Big(-\tfrac{H}{\tau_{fat}}\Big)\Big)\exp\left(\frac{-\frac{1}{3}\Big((A - A_{min} - R)\exp\Big(\frac{H}{\tau_{rec}}\Big) - 500\Big) + H}{\tau_{fat}}\right),$$

$$J_{2,2} = \Big(1 + \tfrac{1}{3\tau_{fat}}\exp\Big(\tfrac{H}{\tau_{rec}}\Big)\Big(R - \gamma\exp\Big(-\tfrac{H}{\tau_{fat}}\Big)\Big)\Big)\exp\left(\frac{-\frac{1}{3}\Big((A - A_{min} - R)\exp\Big(\frac{H}{\tau_{rec}}\Big) - 500\Big) + H}{\tau_{fat}}\right).$$

Linnik P. L. (2018)

Using the spectral property of the Perron-Frobenius operator compute the density of invariant measure by an iteration method:

$$\eta_0 = 1,$$
$$\eta_1 = P\eta_0$$
$$\dots$$
$$\eta_n = P\eta_{n-1} = P^n\eta_0.$$

Density of an invariant measure

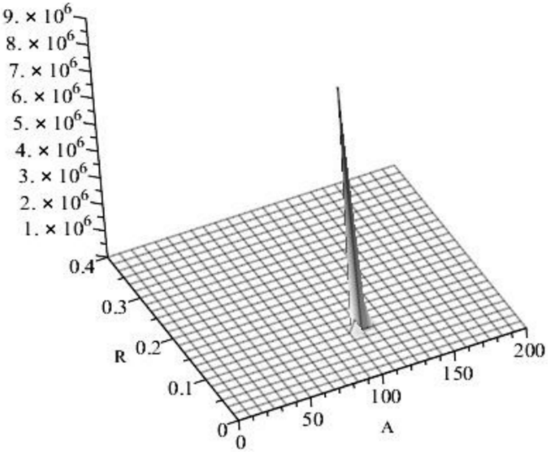

Figure 1: Density of an invariant measure for system (2) ($A_{min} = 33, H = 30, \tau_{rec} = 70, \tau_{fat} = 30, \gamma = 0.3$).

Basic tools of cocycle theory I

Definition 2 (Discrete-time base flow)

Let $(\mathcal{Q},\rho_{\mathcal{Q}})$ be a metric space. A *discrete-time base flow* on $(\mathcal{Q},\rho_{\mathcal{Q}})$ is defined by the mapping $\sigma^{(\cdot)}(\cdot)\colon \mathbb{Z}\times\mathcal{Q}\to\mathcal{Q}\,,(k,q)\mapsto\sigma^k(q)$ satisfying the following properties:

1. $\sigma^0(\cdot)=id_{\mathcal{Q}}$;
2. $\sigma^{k+s}(\cdot)=\sigma^k(\cdot)\circ\sigma^s(\cdot)$ for all $k,s\in\mathbb{Z}$;

Definition 3 (Discrete-time cocycle over the base flow)

Let $(\mathcal{N},\rho_{\mathcal{N}})$ be a metric space. A *discrete-time cocycle over the base flow* $(\{\sigma^k\}_{k\in\mathbb{Z}},\mathcal{Q})$ is defined by the mappings $\{\psi^k(q,\cdot)\}_{\substack{k\in\mathbb{Z}_+,\\ q\in\mathcal{Q}}}$, where the mapping ψ has the folowing properties:

1. $\psi^k(q,\cdot)\colon\mathcal{N}\to\mathcal{N}$ for all $k\in\mathbb{Z}_+$ and all $q\in\mathcal{Q}$;
2. $\psi^0(q,\cdot)=id_{\mathcal{N}}$ for all $q\in\mathcal{Q}$;
3. $\psi^{k+s}(q,\cdot)=\psi^k(\sigma^s(q),\psi^s(q,\cdot))$, for all $k,s\in\mathbb{Z}_+$ and all $q\in\mathcal{Q}$.

Further notation: (σ,ψ).

Definition 4 (Skew-product dynamical system)

Consider the metric space $(\mathcal{W},\rho_{\mathcal{N}})$, where $\mathcal{W}:=\mathcal{Q}\times\mathcal{N}$. A *skew product dynamical system* is a pair $(\{\hat{\psi}^k\}_{k\in\mathbb{Z}_+},(\mathcal{W},\rho_{\mathcal{W}}))$, where $\hat{\psi}^k:\mathcal{W}\to\mathcal{W}$, $\hat{\psi}^k(w):=(\sigma^k(q),\psi^k(q,v))$ for all $w=(q,v)\in\mathcal{W}$ and all $k\in\mathbb{Z}_+$.

Parametrized cocycle generated by a nonautonomous cardiac conduction system

Let us study the parametrized family of skew products:

$$\hat{f}_\alpha : \mathcal{H}_\alpha \times \mathbb{R}^2 \to \mathcal{H}_\alpha \times \mathbb{R}^2, (H, A, R) \mapsto (\sigma_\alpha(H), f_\alpha(H, A, R)), \alpha \in \Lambda, \quad (4)$$

(Λ, ρ_Λ) is a parameter space, $\sigma_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ is the shift map

with the fibre maps

$$f_\alpha(H, A, R) = \begin{cases} f_{1,\alpha}(H, A, R), & \text{for } A < 130, R \in \mathbb{R} \\ f_{2,\alpha}(H, A, R), & \text{for } A \geqslant 130, R \in \mathbb{R}, \end{cases} \quad (5)$$

$$f_{1,\alpha}(H, A, R) = \begin{pmatrix} A_{min} + R\exp\left(-\frac{A+H_0}{\tau_{fat}}\right) + \gamma\exp\left(-\frac{H_0}{\tau_{fat}}\right) + (201 - 0,7A)exp\left(-\frac{H_0}{\tau_{rec}}\right) \\ R\exp\left(-\frac{A+H_0}{\tau_{fat}}\right) + \gamma\exp\left(-\frac{H_0}{\tau_{fat}}\right) \end{pmatrix},$$

$$f_{2,\alpha}(H, A, R) = \begin{pmatrix} A_{min} + R\exp\left(-\frac{A+H_0}{\tau_{fat}}\right) + \gamma\exp\left(-\frac{H_0}{\tau_{fat}}\right) + (500 - 3A)\exp\left(-\frac{H_0}{\tau_{rec}}\right) \\ R\exp\left(-\frac{A+H_0}{\tau_{fat}}\right) + \gamma exp\left(-\frac{H_0}{\tau_{fat}}\right) \end{pmatrix},$$

$$H = (H_0, H_1, H_2, \dots) \in \ell^2(\mathbb{Z}_+; \mathbb{R}) = \mathcal{H}_\alpha. \quad (6)$$

Further notation: $(\sigma_\alpha, f_\alpha)$.

Sun J. et al (1995)

Basic tools of cocycle theory II

Definition 5 (Invariant subsets)

A family of bounded in $\mathcal{N}$ subsets $\hat{\mathcal{Z}} = \{\mathcal{Z}(q)\}_{q\in\mathcal{Q}}$ is said to be *invariant* for the cocycle (τ, ψ) if $\psi^k(q, \mathcal{Z}(q)) = \mathcal{Z}(\tau^k(q))$ for all $k \in \mathbb{Z}_+$ and $q \in \mathcal{Q}$.

Definition 6 (Globally $\mathcal{B}$-pullback attracting subsets)

A family $\hat{\mathcal{Z}} = \{\mathcal{Z}(q)\}_{q\in Q}$ is said to be *globally $\mathcal{B}$-pullback attracting* for the cocycle (τ, ψ) if $\operatorname{dist}(\psi^k(\tau^{-k}(q), \mathcal{B}), \mathcal{Z}(q)) \xrightarrow[k\to\infty]{} 0$ for arbitrary $q \in \mathcal{Q}$ and for any bounded set $\mathcal{B} \subset \mathcal{N}$.

Definition 7 (Global $\mathcal{B}$-pullback attractor)

A family of compact subsets $\hat{\mathcal{A}} = \{\mathcal{A}(q)\}_{q\in Q}$ is called a *global $\mathcal{B}$-pullback attractor* for the cocycle (τ, ψ) if it is invariant and globally $\mathcal{B}$-pullback attracting.

Kloeden P.E., Schmalfuss B. (1997)

Uniform dissipativity and existence of a global $\mathcal{B}$-pullback attractor for the cocycle

Definition 8 (Uniform dissipativity)

We say that the cocycle (τ, ψ) is *uniformly dissipative* if there exists a compact set $\mathcal{D} \subset \mathcal{W}$ and k_0 such, that $\psi^k(q, w) \subset \mathcal{D}$ for all $k \geqslant k_0, k \in \mathbb{Z}_+$, for all $q \in \mathcal{Q}$, for all $w \in \mathcal{W}$, where $\mathcal{D}$ is a dissipativity region of the cocycle (τ, ψ).

Theorem 4

Cocycle $(\sigma_\alpha, f_\alpha)$ is uniformly dissipative, and the dissipativity region $\mathcal{D}_\alpha$ has the following form:

$$\mathcal{D}_\alpha = \left[0, \frac{\eta}{1-3\varepsilon}\right] \times \left[0, \frac{\gamma}{1-\lambda}\right], \text{ where} \tag{7}$$

$\eta = A_{min} + \frac{\gamma}{1-\lambda} + 500\exp(-\frac{H_{min}}{\tau_{rec}})$, $\varepsilon = -\frac{H_{min}}{\tau_{rec}} < \frac{1}{3}$, $\lambda = \exp(-\frac{A_{min}+H_{min}}{\tau_{fat}}) \neq 1$.

Theorem 5

Cocycle $(\sigma_\alpha, f_\alpha)$ has a global $\mathcal{B}$-pullback attractor in the form:

$$\mathcal{A}_\alpha(q) = \bigcap_{k \in \mathbb{Z}_+} \overline{\bigcup_{\substack{s \geqslant k, \\ s \in \mathbb{Z}_+}} f_\alpha^s(\sigma_\alpha^{-s}(q), \mathcal{D}_\alpha)}, \ \forall q \in \mathcal{H}_\alpha, \alpha \in \Lambda, \tag{8}$$

where $\mathcal{D}_\alpha$ is a dissipativity region of cocycle $(\sigma_\alpha, f_\alpha)$.

Global $\mathcal{B}$-attractor for the cocycle generated by the nonautonomous cardiac conduction system

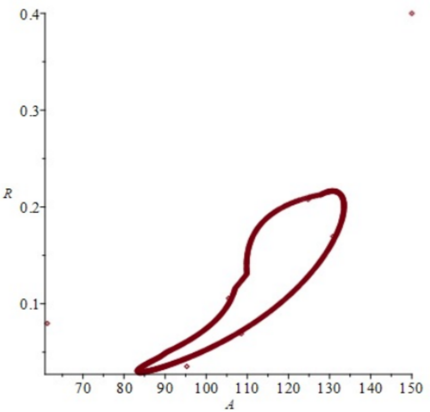

Figure 2: Deterministic forcing.

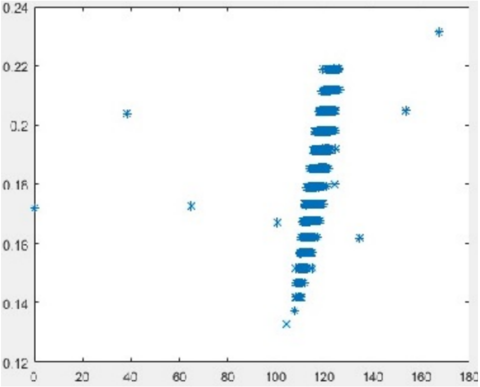

Figure 3: Random forcing with a Poisson distribution.

Measurable cocycles

Let $(\mathcal{Q}, \mathfrak{A}, \mathfrak{m})$ be a probability space.

Definition 9 (Metric dynamical system)

A *metric dynamical system (MDS)* is given by a map $\tau^{(\cdot)}(\cdot)\colon \mathbb{Z} \times \mathcal{Q} \to \mathcal{Q}$ satisfying

1. $\tau^0 = \mathrm{id}_{\mathcal{Q}}$,
2. $\tau^{k+s} = \tau^k \circ \tau^s$, $\forall k, s \in \mathbb{Z}$.

$\{\tau^k\}_{k\in\mathbb{Z}}$ are assumed to be measure preserving, i.e.

$$\tau^k(\mathfrak{m}) = \mathfrak{m}, \forall k \in \mathbb{Z}.$$

Suppose that $(\mathcal{N}, \mathfrak{B})$ is a measurable space.

Definition 10 (Measurable cocycle over the MDS)

A *measurable cocycle over the MDS* $\{\tau^k\}_{k\in\mathbb{Z}}$ is given by a map $\psi : \mathbb{Z}_+ \times \mathcal{Q} \times \mathcal{N} \to \mathcal{N}$ which is for fixed time a $(\mathfrak{A} \otimes \mathfrak{B}, \mathfrak{B})$-measurable mapping and satisfies for all $k, s \in \mathbb{Z}_+$ and almost all $q \in \mathcal{Q}$ and $v \in \mathcal{N}$ the relations

1. $\psi^0(q, v) = v$,
2. $\psi^{k+s}(q, v) = \psi^k(\tau^s(q), \psi^s(q, v))$.

Maltseva A., R. V. (2015), Imkeller, Kloeden (2003)

Definition 11

An *invariant measure* $\hat{\mu}$ for the cocycle (τ, ψ) is a probability measure on $\mathfrak{A} \otimes \mathfrak{B}$ which is invariant w.r.t. the skew product $\{\hat{\psi}^k\}_{k \in \mathbb{Z}_+}$, i.e.

$$\hat{\psi}^k(\hat{\mu}) = \hat{\mu}, \ \forall k \in \mathbb{Z}_+$$

and has the marginal $\pi_{\mathcal{Q}} \hat{\mu} = \mathfrak{m}$ where $\pi_{\mathcal{Q}} : \mathcal{Q} \times \mathcal{N} \to \mathcal{Q}$ is the projection onto $\mathcal{Q}$.

We can characterize invariant measures by their disintegration

$$\hat{\mu}(d(q, v)) = \hat{\mu}_q(dv)\mathfrak{m}(dq) \tag{9}$$

or by

$$\hat{\mu}(\hat{\mathcal{C}}) = \int_{\mathcal{Q}} \hat{\mu}_q(\mathcal{C}_q) d\mathfrak{m}(q), \tag{10}$$

where $\mathcal{C}_q = \{v \in \mathcal{N} \,|\, (q, v) \in \hat{\mathcal{C}}, \ \hat{\mathcal{C}} \in \mathcal{A} \otimes \mathfrak{B}\}$.

Maltseva A., R. V. (2015)

Definition 12

The Perron-Frobenius operator P for the cocycle (τ, ψ) is defined by

$$P\hat{\mu}(q, \mathcal{Z}(q)) := \hat{\mu}(q, \psi^{-1}(q, \mathcal{Z}(\tau(q)))), q \in \mathcal{Q},$$

where $\psi^{-1}(q, \mathcal{Z}(q))$ is the preimage set under $\psi = \psi^1$ of the set $\mathcal{Z}(\tau(q)) \subset \mathcal{N}$.

Approximation of an invariant measure for the cocycle

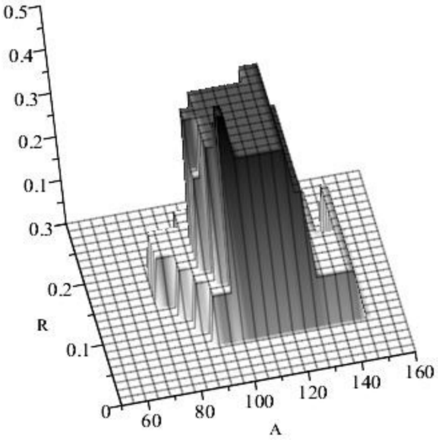

Figure 4: Approximation of an invariant measure on the domain divided into 5×0.025 rectangles.

Let $(\mathcal{Q}_\alpha, \mathfrak{A}_\alpha, \mathfrak{m}_\alpha)$ be a family of probability spaces depending on a parameter $\alpha \in \Lambda$.
The maps $\{\sigma_\alpha^k\}_{k\in\mathbb{Z},\alpha\in\Lambda}$ are assumed to be a measure preserving, i.e. $\sigma_\alpha^k(\mathfrak{m}_\alpha) = \mathfrak{m}_\alpha, k \in \mathbb{Z}, \alpha \in \Lambda$.
Let $\{\hat{\mu}_\alpha\}_{\alpha\in\Lambda}$ be a family of invariant measures for the parametrized skew product, i.e. $\hat{\psi}_\alpha^k(\hat{\mu}_\alpha) = \hat{\mu}_\alpha$ and $\pi_{\mathcal{Q}_\alpha}\hat{\mu}_\alpha = \mathfrak{m}_\alpha$ for $k \in \mathbb{Z}$ and $\alpha \in \Lambda$, where $\pi_{\mathcal{Q}_\alpha} : \mathcal{Q}_\alpha \times \mathcal{M}_\alpha \to \mathcal{Q}_\alpha$ denotes the projection on $\mathcal{Q}_\alpha$.

Definition 13 (Bifurcation point of a family of invariant measures)

A parameter value α_0 is called *bifurcation point of a family of invariant measures* of the family of invariant measures $\{\hat{\mu}_\alpha\}_{\alpha\in\Lambda}$ if this family is not structurally stable at α_0, i.e. if in any neighborhood of α_0 there are parameter values $\alpha \in \Lambda$ such that $\{\hat{\psi}_{\alpha_0}^k\}$ and $\{\hat{\psi}_\alpha^k\}$ are not topologically equivalent.

Maltseva A., R. V. (2015), Arnold L. (1999)

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