Boundedness and finite-time stability for multivalued doubly-nonlinear evolution systems generated by a microwave heating problem

Popov S., Reitmann V.*, Skopinov S.

Department of Applied Cybernetics
Faculty of Mathematics and Mechanics
Saint-Petersburg State University
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1 The two-phase microwave heating problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1-boundary $\partial\Omega$.
Consider the microwave heating problem

$$\begin{cases} \varepsilon(x)E_t(x,t) + \sigma(\theta)E(x,t) = \mathrm{curl}H(x,t), & (x,t) \in Q_T, \\ \mu(x)H_t(x,t) + \mathrm{curl}E(x,t) = 0, & (x,t) \in Q_T, \\ b(\theta(x,t))_t = \nabla[k(x)\nabla\theta(x,t)] + \sigma(\theta)|E(x,t)|^2 & (x,t) \in Q_T, \end{cases} \tag{1}$$

where $T \in \mathbb{R}_+$, $Q_T = \Omega \times [0,T)$, $E(x,t)$ and $H(x,t)$ are the electric and magnetic fields, respectively, $\varepsilon(x), \mu(x)$ and $\sigma(\theta)$ are the electric permittivity, magnetic permeability and electric conductivity, respectively, $b(\theta)$ is the enthalpy operator, $k(x)$ is the thermal conductivity, $\sigma(\theta)|E(x,t)|^2$ is the Joule's heat and

$$b(s) = \begin{cases} b_1(s)\,, & s < \widehat{\theta} \\ [b_1(\widehat{\theta})\,,\; b_2(\widehat{\theta})]\,, & s = \widehat{\theta} \\ b_2(s)\,, & s > \widehat{\theta} \end{cases}$$

is a piecewise smooth function with differentiable monotone increasing functions $b_1(s), b_2(s)$ such that $b_1(\widehat{\theta}) \le b_2(\widehat{\theta})$.

Let $S_T = \partial\Omega \times [0, T)$.

Initial and boundary conditions:

$$\begin{array}{ll} \nu(x) \times E(x,t) = \nu(x) \times G(x,t), & (x,t) \in S_T, \\ \theta(x,t) = 0, & (x,t) \in S_T, \\ E(x,0) = E_0(x), H(x,0) = H_0(x), \theta(x,0) = \theta_0(x), & x \in \Omega, \end{array} \qquad (2)$$

where

- $\nu(x)$ is the outward unit normal on $\partial\Omega$
- $G(x,t)$ is a given external vector function on S_T
- $E_0(x), H_0(x)$ and $\theta_0(x)$ are given functions

Suppose that $\Omega = (0,1)$, $E(x,t) = (0, e(x,t), 0)$ and $H(x,t) = (0,0,h(x,t))$, respectively.
Then we obtain the following system:

$$\begin{cases} \varepsilon(x)e_t(x,t) + \sigma(\theta)e(x,t) = -h_x(x,t), & (x,t) \in (0,1) \times (0,T), \\ \mu(x)h_t(x,t) + e_x(x,t) = 0, & (x,t) \in (0,1) \times (0,T), \\ b(\theta(x,t))_t = k(x)\theta_{xx}(x,t) + \sigma(\theta)e^2(x,t) & (x,t) \in (0,1) \times (0,T). \end{cases} \tag{3}$$

Let us introduce

$$w(x,t) = \int_0^t e(x,\tau)d\tau.$$

Suppose that $\varepsilon(x), \mu(x), k(x) \equiv 1$
Then system (3) becomes

$$\begin{cases} w_{tt} - w_{xx} + \sigma(\theta)w_t = 0, & (x,t) \in (0,1) \times (0,T), \\ b(\theta)_t - \theta_{xx} = \sigma(\theta)w_t^2, & (x,t) \in (0,1) \times (0,T). \end{cases} \tag{4}$$

2 The one-dimensional heating problem

Boundary conditions:

$$w(0,t)=0, w(1,t)=0, \theta_x(0,t)=\theta_x(1,t)=0,\ t\in(0,T).$$

Initial conditions:

$$w(x,0)=0, w_t(x,0)=w_1(x), \theta(x,0)=\theta_0(x),\ x\in(0,1).$$

Assumptions:

(A1) $w_1 \in L^2(0,1)$, θ_0 is nonnegative and $\theta_0 \in L^2(0,1)$.

(A2) $\exists \sigma_0, \sigma_1 > 0$ such that $\sigma_0 \le \sigma(z) \le \sigma_1, \quad z \in [0,\infty)$.

Theorem 1

Suppose **(A1)–(A2)** *are satisfied. Then the system (4) has for any* $T>0$ *a weak solution*
$w \in C^1(0,T;H_0^1(0,1)), \theta \in L^2(0,T;H_0^1(0,1)) \cap C([0,T];L^2(0,1))$.

(Manoranjan, Showalter, Yin, 2006)

Definition 1

A pair of functions $(w(x,t),\theta(x,t))$ is called a **weak solution** of system (19) on the interval $[0, T]$, $T>0$, if $w\in C^1(0, T; H_0^1(\Omega))$, $\theta\in L^2(0, T; H_0^1(\Omega))\cap C(0, T; L^2(\Omega))$ and the following equations are hold

$$\int_0^T\int_0^1[-\varepsilon(x)w_t\psi_t+\frac{1}{\mu(x)}w_x\psi_x+\sigma(\theta)w_t]dxdt=\int_0^1\varepsilon(x)w_1(x)\psi(x,0)dx,$$

$$\int_0^T\int_0^1[-b(\theta)\eta_t+\theta_x\eta_x-\sigma(\theta)w_t^2\eta]dxdt=\int_0^1 b(\theta_0)\eta(x,0),$$

for any test functions $\psi\in L^2(0, T; H_0^1(\Omega))\cap C(0, T; L^2(\Omega)),\forall\eta\in H^1(0, T; H^1(\Omega))$, such that $\psi(x, T)=\eta(x, T)=0,\ \forall x\in\Omega.$

3 Doubly-nonlinear evolutionary system

Let $Y_{1,j}$ and $Y_{2,j}$, $j = 1, 0, -1$ be real Hilbert spaces and $(\cdot,\cdot)_{i,j}$ and $\|\cdot\|_{i,j}$ be scalar products and norms of $Y_{i,j}$, $i = 1, 2$, $j = 1, 0, -1$, respectively. The dense and continuous embeddings $Y_{1,1} \subset Y_{1,0} \subset Y_{1,-1}$ and $Y_{2,1} \subset Y_{2,0} \subset Y_{2,-1}$ are called **rigged Hilbert space structures**.
Consider the system

$$\frac{d}{dt} y_1 = A_1 y_1 + B_1(g_1(z_1) + g_2(z_1, z_2)),\ z_1 = C_1 y_1, \tag{5}$$

$$\frac{d}{dt} \mathbb{B}_2(y_2) = A_2 y_2 + B_2 \phi_2(z_1, z_2),\ z_2 = C_2 y_2, \tag{6}$$

$$y_1(0) = y_{01}, y_2(0) = y_{02}, \tag{7}$$

where $y_i \in Y_{i,1}$, $A_i : Y_{i,1} \to Y_{i,-1}$, $B_i : \Xi_i \to Y_{i,-1}$, $C_i : Y_{i,1} \to Z_i$ are linear bounded operators, $\mathbb{B}_2 : Y_{2,1} \to Y_{2,1}$ is a nonlinear operator, $g_1 : Z_1 \to \Xi_1$, $g_2 : Z_1 \times Z_2 \to \Xi_1$, $\phi_2 : Z_1 \times Z_2 \to \Xi_2$ are nonlinear functions, Ξ_i and Z_i, $i = 1, 2$ are some other Hilbert spaces, $y_{01} \in Y_{1,1}, y_{02} \in Y_{2,1}$.

Let us define the following spaces:
$Y_1 = Y_{1,1} \times Y_{2,1}$, $Y_0 = Y_{1,0} \times Y_{2,0}$, $Y_{-1} = Y_{1,-1} \times Y_{2,-1}$ with scalar products

$$((y_1, w_1), (y_2, w_2))_j = (y_1, y_2)_{1,j} + (w_1, w_2)_{2,j}, \quad j = 1, 0, -1,$$

where $y_1, y_2 \in Y_{1,j}, w_1, w_2 \in Y_{2,j}$, and correspondent norms.
Let $A := (A_1, A_2) : Y_1 \to Y_{-1}$, $B := (B_1, B_2) : \Xi_1 \times \Xi_2 \to Y_{-1}$ and $C := (C_1, C_2) : Y_1 \to Z_1 \times Z_2$ be linear bounded operators,
$\mathbf{B} := (I, \mathbb{B}_2) : Y_1 \to Y_2$ be a nonlinear operator and
$\phi(\cdot, \cdot) := (g_1(\cdot) + g_2(\cdot, \cdot), \phi_2(\cdot, \cdot)) : Z_1 \times Z_2 \to \Xi_1 \times \Xi_2$ be a nonlinear function.
Then system (5) – (7) can be transformed into

$$\frac{d}{dt}\mathbf{B}(y) = Ay + B\phi(z),\ z = Cy, \tag{8}$$

$$y(0) = y_0, \tag{9}$$

where $y = (y_1, y_2)$, $z = (z_1, z_2)$, $y_0 = (y_{01}, y_{02})$.

Let $-\infty \leq T_1 < T_2 \leq +\infty$ be two arbitrary numbers. Let us define in $L^2(T_1, T_2; Y_j)$ the norm $j = 1, 0, -1$

$$\|y\|_{2,j} := \left(\int_{T_1}^{T_2} \|y(t)\|_j^2 \, dt \right)^{1/2} .$$

Let $\mathcal{W}(T_1, T_2; Y_1, Y_{-1})$ be the space of functions y such that $y \in L^2(T_1, T_2; Y_1)$, $\dot{y} \in L^2(T_1, T_2; Y_{-1})$ with the norm

$$\|y\|_{\mathcal{W}(T_1, T_2; Y_1, Y_{-1})} := \left(\|y\|_{2,1}^2 + \|\dot{y}\|_{2,-1}^2 \right)^{1/2} .$$

A **solution** of (8) – (9) is a function $y \in \mathcal{W}(T_1, T_2, Y_1, Y_{-1}) \cap C(T_1, T_2; Y_0)$ satisfing equation (8) – (9) in variational sence, i. e. for a. e. $t \in [T_1, T_2]$ the following equation is satisfied:

$$\left(\frac{d}{dt}\mathbf{B}(y(t)) - Ay(t) - B\phi(z(t)), \eta - y(t) \right)_{-1} = 0,$$

$$\forall \eta \in Y_1, z(t) = Cy(t), y(0) = y_0.$$

Assumptions:

(A3) $Z_1 = \Xi_1 = \Xi_2 = \mathbb{R}$.

(A4) $\exists \kappa_1, \kappa_2,\ \kappa_1 < \kappa_2 : \tilde{\phi}_1(z_1, t) := g_1(z_1) + g_2(z_1, z_2(t))$, where $z_2(t) = C_2 y_2(t)$ and $y_2(t)$ is an arbitrary solution of (5) – (7) such that the following condition is satisfied

$$\kappa_1 z_1^2 \leq \tilde{\phi}_1(z_1, t) z_1 \leq \kappa_2 z_1^2,\ \forall z_1 \in \mathbb{R},\ t \geq 0.$$

(A5) $\exists \kappa_3 > 0 : (\mathbb{B}_2(y_2), A_2 y_2) \leq -\kappa_3 \|y_2\|_{2,1}^2,\ \forall y_2 \in Y_{2,1}$.

(A6) $\exists \kappa_4 > 0$ such that for $\tilde{\phi}_2(t, z_2) = \phi_2(z_1(t), z_2)$ we have $(\mathbb{B}_2(y_2), B_2 \tilde{\phi}_2(t, y_2)) \leq \kappa_4 \|y_2\|_{2,1}^2,\ \forall y_2 \in Y_{2,1}, t \geq 0$.

(A7) System (5) - (7) has a global weak solution.

(A8.1) The operator A_1 in system (5) is **regular**, i. e., for any $T > 0$, $y_{10} \in Y_{1,1}$, $\tilde{y}_{1T} \in Y_{1,1}$, $f_1 \in L^2(0, T; Y_{1,0})$ the solutions of the **direct** problem $\frac{d}{dt} y_1 = A_1 y_1 + f_1(t), y_1(0) = y_{10}$ and the **dual** problem $\frac{d}{dt}\tilde{y}_1 = -A_1^* \tilde{y}_1 + f_1(t), \tilde{y}_1(T) = \tilde{y}_{1T}$ are strongly continuous in the norm of $Y_{1,1}$.

(A8.2) The pair (A_1, B_1) in system (5) is L^2-**controllable**, i. e., for any $y_{10} \in Y_{1,0}$ there exists a control $\xi_1 \in L^2(0, T; Z_1)$ such that the problem $\frac{d}{dt} y_1 = A_1 y_1 + B_1 \xi_1, y_1(0) = y_{10}$ has a solution y_1 for any $T > 0$.

(A8.3) For the **transfer function** $\chi(s) = C_1(A_1 - sI_{Y_{1,1}})^{-1} B_1$ and the Hermitian form:

$$\mathcal{F}(\xi_1, z_1) := \mathrm{Re}(\xi_1 - \kappa_1 z_1)^*(\kappa_2 z_1 - \xi_1),\ \xi_1 \in \mathbb{C}, z_1 \in \mathbb{C}$$

the following frequency domain condition holds

$$\mathrm{Re}(\kappa_1 \chi(i\omega) + I_{\Xi_1})^*(\kappa_2 \chi(i\omega) + I_{\Xi_1}) \geq 0, \quad \forall \omega \in \mathbb{R}.$$

Theorem 2

If conditions **(A3)** – **(A7)** *and* **(A8.1)** – **(A8.3)** *are satisfied then the solutions of system (5) - (7) are bounded on* $(0, \infty)$.

Let us make the following assumptions for system (4):

(A9) $\exists\, a_1 > 0$ such that:

$$|b(z)| \leq a_1|z|, \quad \forall z \in \mathbb{R},\, z \neq \widehat{\theta} \tag{10}$$

(A10) $\exists\, a_2 > 0$ such that:

$$|\sigma(z)| \leq a_2|z|, \quad \forall z \in \mathbb{R}. \tag{11}$$

Corollary 3

Under conditions **(A9)** *and* **(A10)** *all assumptions of Theorem 2 are satisfied. Hence the solutions of system (4) are bounded.*

(Popov, S., R., V., 2014, Popov, S., 2017)

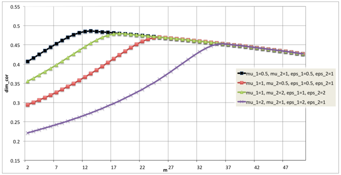
Consider the microwave heating problem in 1-space dimension and without phase-change:

$$\begin{cases} \varepsilon w_{tt} = \frac{1}{\mu} w_{xx} - \sigma(\theta) w_t, & (x,t) \in (0,1) \times (0,T), \\ \theta_t = \theta_{xx} + \sigma(\theta) w_t^2, & (x,t) \in (0,1) \times (0,T), \\ w(0,t) = 0, w(1,t) = 0, & t \in [0,T], \\ \theta(0,t) = \theta(1,t) = 0, & t \in [0,T], \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & x \in (0,1), \\ \theta(x,0) = \theta_0(x), & x \in (0,1). \end{cases} \tag{12}$$

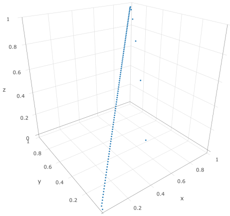
Assumptions:

1) $\mathcal{A}$ is the attractor of the dynamical system generated by the approximation problem to (12);
2) $\varepsilon = 1$, $\mu = 1$ or $\mu = 0.5$;

1) Estimation of the correlation dimension:



2) Embedding by the Takens-Robinson method:



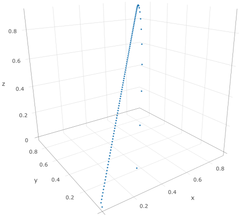


Figure: $\varepsilon = 1$ and $\mu = 0.5$

Figure: $\varepsilon = 1$ and $\mu = 1$

Introduce for $x \in (0,1)$ and $t \in (0,T)$ the functions

$$f(x,t) = f_1(t)(1-x) + f_2(t)x \tag{13}$$

and

$$W(x,t) := w(x,t) - f(x,t), V(x,t) := W_t(x,t) - f_t(x,t). \tag{14}$$

Then the problem (19) becomes

$$\begin{cases} W_t = V - f_t, \\ V_t = W_{xx} - \sigma(\theta)V + f_{tt}, \\ \theta_t - \theta_{xx} = \sigma(\theta)(W_t + f_t)^2, & (x,t) \in (0,1)\times(0,T), \\ W(0,t) = W(1,t) = 0, \quad \theta(0,t) = \theta(1,t) = 0, & t \in (0,T), \\ W(x,0) = W_0(x) := w_0(x) - f(x,0), & x \in (0,1), \\ W_t(x,0) = W_1(x) := w_1(x) - f_t(x,0), & x \in (0,1), \\ \theta(x,0) = \theta_0(x), x \in (0,1). \end{cases} \tag{15}$$

Let us introduce the space $M = H_0^1(0,1) \times L^2(0,1) \times L^1(0,1)$ with norm

$$\|(W,V,\theta)\|_M^2 = \max[\|w_x\|^2_{L^2(0,1)}, \|v\|^2_{L^2(0,1)}, \|\theta\|^2_{L^1(0,1)}]. \qquad (16)$$

Determine the function $y(t,t_0,p) = (W(\cdot,t), V(\cdot,t), \theta(\cdot,t))$ as a solution of the problem (15) with the norm (16). Then (15) can be formally written as system

$$\frac{dy}{dt} = Ay + Bg(V,\theta) + F(t),$$

where $y = (W,V,\theta)$, $F(t) = (-f_t, f_{tt}, 0)$ and A, B are linear operators. If $(W(x,t), V(x,t), \theta(x,t))$ is a solution of (15) we can write it as

$$y(t,t_0,p) = (W(\cdot,t), V(\cdot,t), \theta(\cdot,t)).$$

Definition 2

System (15) is called (α, β, t_0, T')-**stable**, where $0 < \alpha \leq \beta$, $t_0 > 0$ and $T' \geq 0$ are nonnegative numbers, if from the inequality $||y(t_0)||_Y < \alpha$ it follows that $||y(t)||_Y < \beta$ for all $t \in [t_0, t_0 + T')$.

(Weiss, Infante, 1965) - ODE-system.
(Chetaev, 1960) - visco-elastic systems.

(A14) Consider the heat equation:

$$\begin{cases} \theta_t - \theta_{xx} = 0, \\ \theta(x,0) = \theta_0(x), \quad x \in (0,1), \\ \theta(0,t) = \theta(1,t) = 0, \quad t \in (0,T). \end{cases} \tag{17}$$

Let c_D be the upper bound of $\theta(x,t)$ for $x \in (0,1), t \in (0,T)$, where $\theta(x,t)$ is an arbitrary solution of system (17).

(A15) $|N(t)| \leq c_N$ for any $t \in (0,T)$, where $N(t) := \int_0^t \sum_{i=1}^2 [f_{it} + |f_{itt}|] d\tau$. Here f_{it} and f_{itt} are defined as

$$f_{it} = \frac{df_i}{dt}, \quad f_{itt} = \frac{d^2 f_i}{dt^2}. \tag{18}$$

4 Finite-time stability for non-autonomous heating problem

Consider the one-dimensional microwave heating problem with non-autonomous boundary conditions:

$$\begin{cases} w_{tt} - w_{xx} + \sigma(\theta) w_t = 0, & (x,t) \in (0,1) \times (0,T), \\ \theta_t - \theta_{xx} = \sigma(\theta) {w_t}^2, & (x,t) \in (0,1) \times (0,T), \\ w(0,t) = f_1(t)\,,\ w(1,t) = f_2(t), & t \in (0,T), \\ \theta(0,t) = \theta(1,t) = 0, & t \in (0,T), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), & x \in (0,1), \\ \theta(x,0) = \theta_0(x), & x \in (0,1), \end{cases} \tag{19}$$

where $\theta(x,t)$ is the temperature, $w(x,t)$ is the variable, determining the electric field, f_1 and f_2 are given functions.

Let the following conditions are satisfied:
(A11) There exists constants σ_0 and σ_1, such that
$0 < \sigma_0 \leq \sigma(\theta) \leq \sigma_1(1+\theta), \quad \forall\theta > 0;$
(A12)σ is locally Lipschitz on $(0, +\infty)$,
(A13)$f_1, f_2 \in C^2(\mathbb{R}), \quad f_1(0) = 0, f_2(0) = 0, \quad w_t(x, 0), \theta_0(x) \in L^2(0, 1)$.
Denote $v := w_t$.

Theorem 4

There exists a global weak solution $(w(x,t), \theta(x,t))$ of the problem (19) such that $w, v \in C([0, T]; L^2(0,1))$; $\theta \in L^2(0, T; H^1(0,1))$.

(Yin, 1998)

Theorem 5

Consider problem (15) and let the conditions (A11)-(A15) be satisfied. Then system (15) is $(\alpha, \beta, 0, T)$-stable, *if for the given parameters* $\alpha > 0, t_0 = 0, T > 0$ *the parameter* β *is calculated by*

$$\beta = \max[\beta_1, \beta_2], \quad \textit{where} \tag{20}$$

$$\beta_1 = 4c_D \max\left[\sigma_1, \frac{1}{\sigma\, 0}\right] c_N + 2c_D \max\left[\sigma_1, \frac{1}{\sigma\, 0}\right] c(f, T) + c_D\alpha + 4c_D(c(f, T) + c_D\alpha)(c_N + c_D\alpha)c(f, T), \tag{21}$$

$$\beta_2 = \sqrt{\max\left[\sigma_1, \frac{1}{\sigma\, 0}\right] c_N + c(\delta)\sigma_1(c_N + c_D\alpha)c(f, T)}. \tag{22}$$

where $f(t) := \sum_{i=1}^{2} |f_{it}|$, $c(f, T) = e^{\int_0^T f(\tau)d\tau} \int_0^T f(\tau)e^{-\int_0^\tau f(\eta)d\eta} d\tau$.

(Skopinov, S., 2017)

Introduce the family of mappings

$$\varphi^{(\cdot)}(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R} \times M \to M \qquad \text{by}$$
$$\varphi^t(t_0, p) = y(t + t_0, t_0, p)$$

for any $t \in \mathbb{R}_+, t_0 \in \mathbb{R}_+, p \in M$, where M is the Banach space $M = H_0^1(0,1) \times L^2(0,1) \times L^2(0,1)$ with the norm

$$\|(W, V, \theta)\|_M^2 = \|W\|_{H_0^1}^2 + \|V\|_{L^2}^2 + \|\theta\|_{L^2}^2.$$

The mapping $\varphi^{(\cdot)}(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R} \times M \to M$ is said to be a **process** if the following conditions are satisfied:

1) $\varphi^0(s,\cdot) = I_M$ for all $s \in \mathbb{R}_+$;
2) $\varphi^{t_1+t_2}(s,p) = \varphi^{t_1}(s + t_2, \varphi^{t_2}(s,p))$ for all $(s,p) \in \mathbb{R} \times M$ and $t_1, t_2 \in \mathbb{R}_+$.

Examples of processes are **dynamical systems** for which $\varphi^t(s,\cdot) = \varphi^t(\cdot)$ for $s \in \mathbb{R}_+$ and $t \in \mathbb{R}$.
Suppose $(\tau, u_\tau) \in \mathbb{R} \times M$. The mapping $\mathbb{R}_+ \ni t \mapsto u(t) \in M$ is said to be a **motion** of the process $(\{\varphi^t(s,\cdot)\}_{\substack{t\in\mathbb{R}_+ \\ s\in\mathbb{R}}}, (M, \rho_M))$ through u_τ for $t = 0$ if $u(t) = \varphi^t(\tau, u(\tau)), \forall t > 0$, and $u(0) = u_\tau$.
Assume that $0 < \alpha \leq \beta$ and $T' > 0$, $t_0 \geqslant 0$, are numbers and $p \in M$ is a fixed point. The process $(\{\varphi^t(s,\cdot)\}_{\substack{t\in\mathbb{R}_+ \\ s\in\mathbb{R}}}, (M, \rho))$ is said to be $(\alpha, \beta, t_0, T', p)$ stable if the inequality $\rho_M(\varphi^0(\tau, u_\tau), p) < \alpha$ for an arbitrary pair $(\tau, u_\tau) \in \mathbb{R}_+ \times M$ implies that $\rho_M(\varphi^t(\tau, u_\tau), p) < \beta$ for all $t \in [t_0, t_0 + T')$.
Suppose $(\{\varphi^t(s,\cdot)\}_{\substack{t\in\mathbb{R}_+ \\ s\in\mathbb{R}}}, (M, \rho_M))$ is a process. The map $\phi : \mathbb{R} \times M \to \mathbb{R}$ is said to be a **Lyapunov functional** for this process if the following conditions are satisfied:

(A16) The family of maps $\phi(t,\cdot): M \to \mathbb{R}$ is continuous;

(A17) For arbitrary $t \in \mathbb{R}$ and $u \in M$ there exists the limit

$$\dot{\phi}(t,u) := \lim_{s \to 0+} \sup \frac{1}{s}[\phi(t+s, \varphi^s(t,u)) - \phi(t,u)].$$

Theorem 6

Suppose that $(\{\varphi^t(s,\cdot)\}_{\substack{t\in\mathbb{R}_+\\ s\in\mathbb{R}}}$ *is a process,* $I := [t_0, t_0 + T']$ *is a time interval,* $0 < \alpha \le \beta$, $T' > 0$, $t_0 > 0$ *are positive numbers,* $u_\tau \in M, p \in M$ *are some points and there exist a Lyapunov functional* $\phi : I \times M \to \mathbb{R}$ *for the process and an integrable function* $g : I \to \mathbb{R}$ *such that:*

(i) $\dot{\phi}(t, u(t)) < g(t)$ *for arbitrary* $t \in I$ *and arbitrary functions* $u(\cdot) \in C(t_0, t_0 + T', M)$ *such that* $\alpha \le \rho_M(u(t), p) \le \beta$ *for all* $t \in I$;

(ii) $\int_s^t g(\tau)d\tau \le \min_{u\in M:\rho_M(u,p)=\beta} \phi(t,u) - \max_{u\in M:\rho_M(u,p)=\alpha} \phi(s,u)$ *for all* $s, t \in I, s < t$. *Then the process* $(\{\varphi^t(s,\cdot)\}_{\substack{t\in\mathbb{R}_+\\ s\in\mathbb{R}}}, (M, \rho_M))$ *is* $(\alpha, \beta, t_0, T', p)$*-stable.*

Consider problem (15). Initial and boundary conditions:

$$\sigma(\theta) = 0.2(1+\theta), \theta \in \mathbb{R}, w_0(x) = 0, w_1(x) = 0, \theta_0(x) = 0, x \in (0,1),$$
$$f_1(t) = f_2(t) = 2\sin 2t, t \in \mathbb{R}. \tag{23}$$

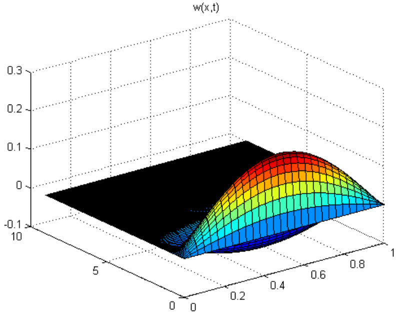


Figure: 1 The solution component $w(x,t)$

6 Numerical results for the one-dimensional heating problem

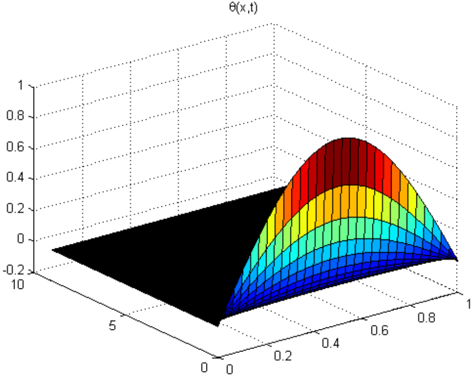


Figure: 2 The solution component $\theta(x, t)$

7 Multivalued dynamical systems

Suppose (M, ρ_M) is a complete metric space. Let $\varphi^t : M \to 2^M, \forall t \in \mathbb{R}_+$, be a family of maps. The pair $(\{\varphi^t\}_{t\in\mathbb{R}_+}, (M, \rho_M))$ is said to be a **multivalued dynamical system** (MDS) if the following conditions are satisfied:

1) $\varphi^0(p) = \{p\}, \quad \forall p \in M,$
2) $\varphi^{t_1+t_2}(p) \subset \varphi^{t_1}(\varphi^{t_2}(p)), \quad \forall t_1, t_2 \in \mathbb{R}_+, \forall p \in M\,.$

The MDS $(\{\varphi^t\}_{t\in\mathbb{R}_+}, (M, \rho_M))$ is said to be **continuous with respect to the initial conditions** if for arbitrary sequences $\{t_n\} \subset \mathbb{R}_+, \{p_{n0}\} \subset M$ such that $t_n \to t, p_{n0} \to p_0$ as $n \to \infty$ for some $t \in \mathbb{R}_+$ and $p_0 \in M$ there exists for any $n \in \mathbb{N}$ a $\tilde{p}_n \in M$ satisfying $\tilde{p}_n \in \varphi^{t_n}(p_{n0})$ and $\tilde{p}_n \to \tilde{p}$ as $n \to \infty$.

A subset $Z \subset M$ is said to be

- **attracting** if $\mathrm{dist}(\varphi^t(p), Z) \to 0$ as $t \to \infty, \quad \forall p \in M$, where $\mathrm{dist}(W, W') = \inf_{p \in W, q \in W'} \rho_M(p, q), W, W' \subset M$,
- **absorbing** if $\forall p \in M \; \exists T \in \mathbb{R}_+ : \forall t > T, t \in \mathbb{R}_+, \varphi^t(p) \subset Z$,
- **invariant** if $\varphi^t(Z) = Z, \quad \forall t \in \mathbb{R}_+$,
- a **global attractor** if Z is bounded and closed, invariant and globally attracting.

Let us consider the 3 D heating problem. Introduce the set

$$D = \{(E, H, \theta) \in H_0(\mathrm{curl}, \Omega) \times (H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega)) \times H_0^1(\Omega);$$
$$\mu H \in \mathbb{H}_1(\Omega)^\perp \cap H(\mathrm{div0}, \Omega)\}, \mathbb{H}_1(\Omega) = H(\mathrm{curl0}, \Omega) \cap H_0(\mathrm{div0}, \Omega), \quad (24)$$

with the norm $\|(E, H, \theta)\|_D := \max\{\|E\|_{L^2(\Omega)^3}, \|H\|_{L^2(\Omega)^3}, \|\theta\|_{L^2(\Omega)}\}$.

Here

$$\begin{cases} H(curl,\Omega) = \{v \in L^2(\Omega)^3 : curl\ v \in L^2(\Omega)^3\}, \\ H(div,\Omega) = \{v \in L^2(\Omega)^3 : div\ v \in L^2(\Omega)^3\}, \\ H_0(curl,\Omega) = \{v \in H(curl,\Omega) : v \times \nu = 0, \forall v, \nu \in \partial\Omega\}, \\ H_0(div,\Omega) = \{v \in H(div,\Omega) : v \cdot \nu = 0, \forall v, \nu \in \partial\Omega\}, \\ H(div0,\Omega) = \{v \in L^2(\Omega)^3 : div\ v = 0\}, \\ H_0(div0,\Omega) = H_0(div,\Omega) \cap H(div0,\Omega). \end{cases} \tag{25}$$

Introduce the map

$$\varphi : \mathbb{R}_+ \times D \to 2^D \tag{26}$$

through $\varphi^t(E_0, H_0, \theta_0) = \{(\tilde{E}, \tilde{H}, \tilde{\theta}) \in D : \exists$ solution (E, H, θ) of (1) with initial values E_0, H_0, θ_0 and $E(\cdot, t) = \tilde{E}, H(\cdot, t) = \tilde{H}, \theta(\cdot, t) = \tilde{\theta}\}$.

Theorem 7

Consider the map (26). Then:

1) *(26) defines a MDS;*
2) *The MDS (26) is continuous with respect to the initial conditions;*
3) *The MDS (26) has the global attractor* $A = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \varphi^t(B_0)}$, *where* B_0 *is a compact absorbing set for (26).;*

(Zyryanov, R., 2017)

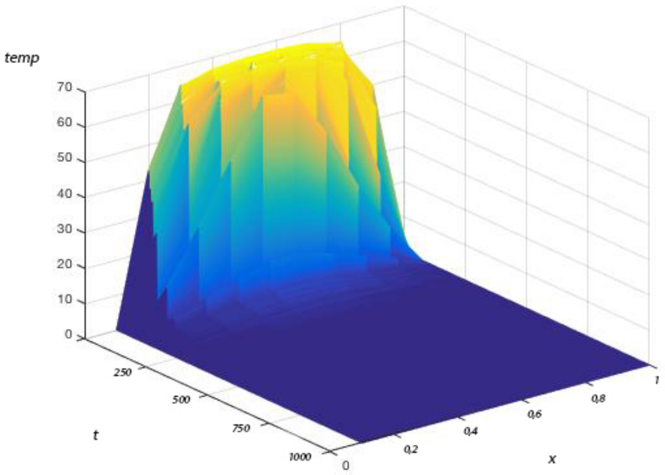


Figure: 3 Change of the temperature at the line $x \in (0,1), y = 0.5, z = 0.5$

8 Numerical results for multivalued dynamical systems

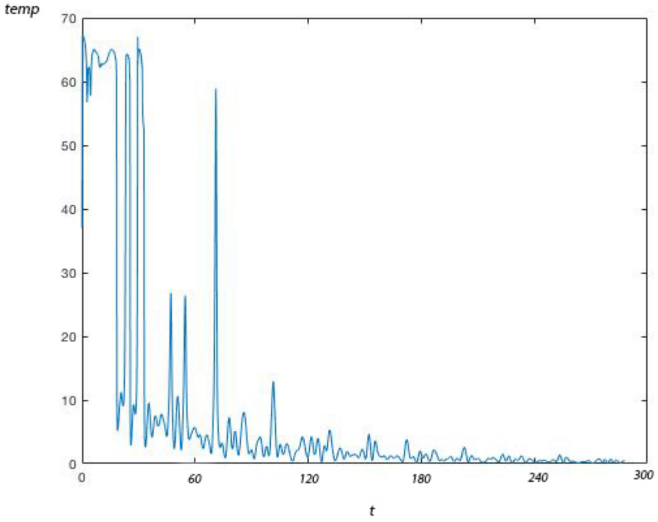


Figure: 4 Change of the temperature at a central point inside the cube

References

1) Kalinichenko D.Yu., Reitmann V. and Skopinov S.N. *Asymptotic behavior of solutions to a coupled system of Maxwell's equations and a controlled differential inclusion*, Discrete and Continuous Dynamical Systems, Supplement, pp. 407-414, 2013
2) Reitmann V. and Skopinov S.N. *On a finite time interval stability for the one-dimensional microwave heating problem*, Vestnik SPbGU, 1, 2(60), pp. 54-59, 2014
3) Yin H.-M. *On Maxwell's equations in an electromagnetic field with the temperature effect*, SIAM J. on Mathematical Analysis, 15(4), Vol. 29., pp. 637-651, 1998
4) Weiss L., Infante E.F., *On the stability of systems defined over a finite time interval*, Proc. Nat. Acad. Sci, 54, pp. 44-48, 1965
5) Chetaev N.G., *About some questions, relating to the problem of the stability of unsteady motions*, PMM24, 1, pp. 16-19, 1960

6) A. Likhtarnikov and V. Yakubovich, *The frequency theorem for equations of evolutionary type*, Siberian Math. J., 17(5), pp.790–803, 1976
7) Manoranjan V. S., Showalter R., Yin H.-M. *On two-phase Stefan problem arising from a microwave heating process*, Discrete and Cont. Dyn. Sys. Series A., 15(4), pp. 1155–1168, 2006
8) Popov, S. A., Reitmann, V., *Frequency domain conditions for finite dimensional projectors and determining observations for the set of amenable solutions*, Discrete Contin. Dyn. Syst., 34(1), pp. 249-267, 2014
9) Popov, S. A, *Localization of invariant sets and attractors of evolutionary systems arising from one- and two-phase heating problems and their numerical reconstruction by the Takens method*, Autoreferat (to appear), 2017

References

10) Zyryanov, D. A., Reitmann, V., *Attractors in multivalued dynamical systems for the two-phase heating problem*, Electr. J. Differential Equations and Control Processes (to appear)

11) Skopinov, S. N, *Lyapunov function method for analysis of the finite time stability of the multivalued heating processes*, Autoreferat (to appear), 2017