Global $\mathcal{B}$-Pullback Attractors for Cocycles Generated by Discrete-Time Cardiac Conduction Models

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Basic Tools of Cocycle Theory I

Definition 1 (Discrete-time base flow)

Let Q be a topological space. A *discrete-time base flow* on Q is defined by the mapping $\tau^{(\cdot)}(\cdot) \colon \mathbb{Z} \times Q \to Q\,, (k,q) \mapsto \tau^k(q)$ satisfying the following properties:

1. $\tau^0(\cdot) = id_Q$;
2. $\tau^{k+j}(\cdot) = \tau^k(\cdot) \circ \tau^j(\cdot)$ for all $k, j \in \mathbb{Z}$;

Definition 2 (Discrete-time cocycle over the base flow)

Let (N, ρ_N) be a metric space. A *discrete-time cocycle over the base flow* $(\{\tau^k\}_{k\in\mathbb{Z}}, Q)$ is defined by the mappings $\{\psi^k(q,\cdot)\}_{\substack{k\in\mathbb{Z},\\ q\in Q}}$, where the mapping ψ has the folowing properties:

1. $\psi^k(q,\cdot) \colon N \to N$ for all $k \in \mathbb{Z}$ and all $q \in Q$;
2. $\psi^0(q,\cdot) = id_N$ for all $q \in Q$;
3. $\psi^{k+j}(q,\cdot) = \psi^k(\tau^j(q), \psi^j(q,\cdot))$, for all $k, j \in \mathbb{Z}$ and all $q \in Q$.

Further notation: (τ, ψ).

Definition 3 (Invariant subsets)

A family of bounded in N subsets $\hat{\mathcal{Z}} = \{\mathcal{Z}(q)\}_{q \in Q}$ is said to be *invariant* for (τ, ψ) if $\psi^k(q, \mathcal{Z}(q)) = \mathcal{Z}(\tau^k(q))$ for all $k \in \mathbb{Z}$ and $q \in Q$.

Definition 4 (Globally $\mathcal{B}$-pullback attracting subsets)

A family $\hat{\mathcal{Z}} = \{\mathcal{Z}(q)\}_{q \in Q}$ is said to be *globally $\mathcal{B}$-pullback attracting* for (τ, ψ) if $dist(\psi^k(\tau^{-k}(q), \mathcal{B}), \mathcal{Z}(q)) \xrightarrow[k \to \infty]{} 0$ for arbitrary $q \in Q$ and for any bounded set $\mathcal{B} \subset N$.

Definition 5 (Global $\mathcal{B}$-pullback attractor)

A family of compact subsets $\hat{\mathcal{A}} = \{\mathcal{A}(q)\}_{q \in Q}$ is called a *global $\mathcal{B}$-pullback attractor* for (τ, ψ) if it is invariant and globally $\mathcal{B}$-pullback attracting.

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[1] Kloeden P.E., Schmalfuss B. *Nonautonomous systems, cocycle attractors and variable time-step discretization*, Numerical Algorithms, 1997, vol. 14, № 1-3, p. 141-152.

Control System with Disturbances

Suppose that

- $(Y, (\cdot,\cdot)_Y)$, $(\Xi, (\cdot,\cdot)_\Xi)$ and $(Z, (\cdot,\cdot)_Z)$ are Hilbert spaces,
- $A \in \mathcal{L}(Y, Y)$, $B \in \mathcal{L}(\Xi, Y)$, $C \in \mathcal{L}(Y, Y)$ are linear bounded operators,
- $\phi\colon \mathbb{Z} \times Z \to \Xi$ is a nonlinear function,
- $\{\zeta_k\}_{k\in\mathbb{Z}}$ is a sequence in Y.

Consider the discrete time control system with disturbances

$$y_{k+1} = Ay_k + B\xi_k + \zeta_k, \tag{1}$$

where $\xi_k = \phi(k, z_k)$, and $z_k = Cy_k, k \in \mathbb{Z}$.
The linear part of (1) is given by

$$y_{k+1} = Ay_k + B\xi_k, \tag{2}$$

where $\{\xi_k\}_{k\in\mathbb{Z}}$ is some sequence in Ξ.

ℓ^2-controllability

Suppose H is a Hilbert space with associated norm $||\cdot||_H$.
Introduce the space of square summable sequences

$$\ell^2(\mathbb{Z}; H) := \left\{ h = \{h_k\}_{k\in\mathbb{Z}} \subset H \;\middle|\; ||h||^2_{\ell^2(\mathbb{Z};H)} := \sum_{k=-\infty}^{\infty} ||h_k||^2_H < \infty \right\}. \quad (3)$$

Definition 6 (ℓ^2-controllability)

System (2) and the pair (A, B) are called ℓ^2*-controllable* if for any $y_0 \in Y$ there exists a control $\{\xi_k(y_0)\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \Xi)$ such that for the solution $y_k(0, y_0)$ of (2) with this control we have $\{y_k(0, y_0)\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}; Y)$.

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[2] Antonov V.G., Likhtarnikov A.L., Yakubovich V.A. *A discrete frequency theorem for the case of Hilbert spaces of states and controls. I*, Vestnik Leningr.Univ.Math, 1980, vol. 8. p.1–11

Consider the system

$$y_{k+1} = Ay_k + B\xi_k + \zeta_k, k \in \mathbb{Z}. \qquad (1)$$

Let us derive frequency domain conditions for dissipativity of system (1).

Definition 7 (Dissipativity)

We say that system (1) is *dissipative* if there exists a compact set $\mathcal{D} \subset Y$ such that for any solution $y_k(k_0, y_0)$ of (1) with $y_{k_0}(k_0, y_0) = y_0$ there exists a time $\bar{k} = \bar{k}(k_0, y_0)$ such that $y_k(k_0, y_0) \in \mathcal{D}$ for all $k \geqslant \bar{k}$.

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[3] Dmitriev, Yu. A. *Frequency conditions for the dissipativity and for the existence of periodic solutions in pulse systems with one non-linear block*, Dokl. Akad. Nauk SSSR, 1965, vol. 164, no. 1, p.28–31 (in Russian)

Quadratic Form

Suppose that there is given the quadratic form on $Y \times \Xi$

$$\mathcal{F}(y,\xi) := (F_1 y, y)_Y + 2Re(F_2\xi, y)_Y + (F_3\xi,\xi)_\Xi$$

with $F_1 = F_1^* \in \mathcal{L}(Y,Y), F_2 \in \mathcal{L}(\Xi, Y)$ and $F \in \mathcal{L}(\Xi,\Xi)$.
According to the quadratic form $\mathcal{F}$ we assume the following property

$$\mathcal{F}(y,\xi) \geqslant 0 \text{ for all } y \in Y \text{ and } \xi = \phi(k, Cy), k \in \mathbb{Z}. \tag{4}$$

Consider the Hermitian extension $\mathcal{F}^c$ of $\mathcal{F}$ given on $Y^c \times \Xi^c$ and coinciding with $\mathcal{F}$ on $Y \times \Xi$.

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[4] Pankov A.A. *Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations*, Kluwer Academic Publishers, London, 1990

Discrete Frequency Theorem

Consider the system

$$y_{k+1} = Ay_k + B\xi_k + \zeta_k, k \in \mathbb{Z}. \quad (1)$$

Theorem 1

Suppose that the following conditions are satisfied:

1. *The pair (A, B) is ℓ^2-controllable;*
2. *The spectrum $\sigma(A)$ of A lies inside the unit disc in $\mathcal{C}$, i.e.* $\sigma(A) \subset \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$;
3. *The frequency domain condition is satisfied, i.e. there exist some $\delta > 0$ such that*

$$\mathcal{F}^c((\lambda I_y - A)^{-1}B\xi, \xi) \leqslant -\delta||\xi||^2_\Xi \; \forall \xi \in \Xi, \forall \lambda \in \mathbb{C}, |\lambda| = 1;$$

4. *Inequality (4) is satisfied uniformly with respect to k.*

Then system (1) has a bounded on $\mathbb{Z}$ solution $\{\bar{y}_k\}_{k\in\mathbb{Z}}$. If the function $\phi(k, z)$ and the sequence $\{\zeta\}_k$ are almost periodic then the solution $\{\bar{y}_k\}$ is also almost periodic. Suppose ϕ does not depend on k and $\{\zeta_k\}_{k\in\mathbb{Z}}$ is an ergodic or mixing process. Then $\{\bar{y}_k\}$ is ergodic o mixing.

Uniqueness of the Solution

Consider again the system

$$y_{k+1} = Ay_k + B\xi_k + \zeta_k, k \in \mathbb{Z}. \tag{1}$$

Theorem 2

Suppose that the conditions $1) - 3)$ *of Theorem 1 and the following condition are satisfied:*

1. $\phi(k, 0)$ *is bounded on* $\mathbb{Z}$*;*
2. $\mathcal{F}(y - y', \xi - \xi') \geqslant 0$ *for all* $y, y' \in Y$ *and* $\xi = \phi(k, Cy)$*,* $\xi' = \phi(k, Cy')$*,* $k \in \mathbb{Z}$*.*

Then the solution $\{\bar{y}_k\}_{k\in\mathbb{Z}}$ *from Theorem 1 is unique and there exists constants* $c_1 > 0$ *and* $\rho \in (0, 1)$ *such that for any other solution* $\{y_k\}_{k\in\mathbb{Z}}$ *of (1) we have*

$$||y_k - \bar{y}_k||_Y \leqslant c_1\rho^{k-k_0}||y_{k_0} - \bar{y}_{k_0}||_Y, k \geqslant k_0.$$

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[5] Maltseva A.A., Reitmann V. *Global stability and bifurcations of invariant measures for the discrete cocycles of the cardiac conduction system's equations*, Differential Equations, vol. 50, 2014, p. 1718-1732

Idea of the Proof of the Theorem (2)

Consider the system

$$y_{k+1} = Ay_k + B\xi_k + \zeta_k, k \in \mathbb{Z}. \tag{1}$$

From [2] it follows that there exists an operator $P = P^* \in \mathcal{L}(Y, Y)$ with $P \gg 0$ such that the Lyapunov function $V(y) := (y, Py)_Y, y \in Y$ satisfied the inequality

$$V(Ay + B\phi(k, y) + \zeta_k) - V(y) \leqslant -\varepsilon||y||_Y^2, \forall y \in Y, \forall k \in \mathbb{Z}.$$

Along with (1) consider the equation

$$y_{k+1}^{(q)} = Ay_k^{(q)} + B\phi^{(q)}(k, z_k) + \zeta_k^{(q)}, z_k = Cy_k^{(q)}, k \in \mathbb{Z}, q \in \mathbb{Z}_B, \tag{5}$$

where $\mathbb{Z}_B$ is the Bohr compactification of the group $\mathbb{Z}$,

$$\phi^{(q)}(k, z_k) = \hat{\phi}(q + k, z), q \in \mathbb{Z}_B, k \in \mathbb{Z}, z \in Z,$$

where $\hat{\phi}(\cdot, \cdot)$ is the continious extension of ϕ on $\mathbb{Z}_B \times Z$.

Example: Cardiac Conduction Model

Consider following system:

$$\begin{cases} A_{k+1} = A_{min} + R_k exp\Big(-\frac{A_k+H_k}{\tau_{fat}}\Big) + \gamma exp\Big(-\frac{H_k}{\tau_{fat}}\Big) + \beta_k exp\Big(-\frac{H_k}{\tau_{rec}}\Big)\Big), \\ R_{k+1} = R_k exp\Big(-\frac{A_k+H_k}{\tau_{fat}}\Big) + \gamma exp\Big(-\frac{H_k}{\tau_{fat}}\Big), \end{cases} \quad (6)$$

where:

- $$\beta(A_k) := \beta_k = \begin{cases} 201 - 0.7A_k, \text{ for } A_k < 130, \\ 500 - 3A_k, \text{ for } A_k \geqslant 130; \end{cases}$$
- $A_{min}, \tau_{rec}, \gamma, \tau_{fat}$ are positive constants, $k \in \mathbb{Z}$;
- $(A, R) \in \mathbb{R}^2$;
- A_k is the conduction time of kth impulse;
- H_k is the nodal recovery time during cycle k.
- R_k is the drift in the nodal conduction time of kth impulse.

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[6] Sun J., Amellal F., Glass L., Billete J. *Alternans and period-doubling bifurcations in atrioventricular nodal conduction*, J. theor. Biol., 1995, 173,p. 79-91.

Example: Cardiac Conduction Model. Dissipativity

Consider the system:

$$\begin{cases} A_{k+1} = A_{min} + R_k exp\Big(-\frac{A_k+H_k}{\tau_{fat}}\Big) + \gamma exp\Big(-\frac{H_k}{\tau_{fat}}\Big) + \beta_k exp\Big(-\frac{H_k}{\tau_{rec}}\Big)\Big), \\ R_{k+1} = R_k exp\Big(-\frac{A_k+H_k}{\tau_{fat}}\Big) + \gamma exp\Big(-\frac{H_k}{\tau_{fat}}\Big). \end{cases} \tag{6}$$

Theorem 3

System (6) is dissipative, and the dissipative set $\mathcal{D}$ has the following form:

$$\mathcal{D} = \Big[A_{min}, d_1\frac{c_2}{1-c_2}\Big] \times \Big[0, d_2\frac{c_3}{1-c_3}\Big], \text{ where}$$

$0 < 3exp\Big(-\frac{H_k}{\tau_{rec}}\Big) \leqslant c_2$, $0 < exp\Big(-\frac{A_k+H_k}{\tau_{fat}}\Big) \leqslant c_3 < 1$, $\gamma exp\Big(-\frac{H_k}{\tau_{fat}}\Big) \leqslant d_2$,
$A_{min} + d_2\frac{c_3}{1-c_3} + 500exp\Big(-\frac{H_k}{\tau_{rec}}\Big) \leqslant d_1$.

Suppose that

- Q has the structure of a measurable space $(Q, \mathfrak{C}, \nu)$ with measure ν;
- $\mathfrak{B}$ is the σ-algebra of Borel subsets of N.

Definition 8 (Invariant measure)

A measure ν on $\mathfrak{C} \times \mathfrak{B}$ is called *invariant measure* for the cocycle (τ, ψ) if

$$\mu(\varphi^{-1}(\mathcal{A})) = \mu(\mathcal{A}), \forall \mathcal{A} \in \mathfrak{C} \times \mathfrak{B}, k \in \mathbb{Z},$$

where $\varphi : Q \times N \to Q \times N$ is defined by $\varphi^k(q, v) = (\tau^k(q), \psi^k(q, v))$.

The disintegrations of ν are given by the family of measures μ_q on $\mathfrak{B}$ satisfying

$$\nu(\mathcal{A}) = \int_Q \mu_q(\mathcal{A}_q) d\nu(q). \tag{7}$$

[7] Maltseva, A., Reitmann, V., Bifurcations of invariant measures in discrete-time parameter dependent cocycles, *Mathematica Bohemica*, 2015, no. 2, vol. 140, pp. 205–213.