Stability and bifurcation investigation of discrete-time nonlinear systems by realization theory methods

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Equadiff 2015

July 06 - 10, 2015, Lyon, France

Consider the nonlinear Volterra integral equation

$$\sigma(t) = h(t) + \int_0^t G(t-s)\varphi(\sigma(s), s)\,\mathrm{d}s \qquad (1)$$

with $\sigma : \mathbb{R}_+ \to U$ ($\mathbb{R}^n$, Hilbert space) as output,
$h : \mathbb{R}_+ \to U$ as perturbation,
$u(\cdot) := \varphi(\sigma(\cdot), \cdot) : \mathbb{R}_+ \to U$ as control,
$\forall t \geq 0 : G(t) \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ as kernel and
$\varphi : U \times \mathbb{R}_+ \to U$ as nonlinearity .

(A1) $t \mapsto G(t)$ is twice piecewise-differentiable and
$\exists\, c > 0,\ \rho_0 > 0,\ \lambda > 0 : \|G(t)\|_{\mathcal{L}(\mathcal{U},\mathcal{U})} \leq c e^{-\rho_0 t}, \quad \forall t \geq 0,$
$\int_0^\infty \left(\|\dot{G}(t)\|^2_{\mathcal{L}(\mathcal{U},\mathcal{U})} + \|\ddot{G}(t)\|^2_{\mathcal{L}(\mathcal{U},\mathcal{U})} \right) e^{2\lambda t}\,\mathrm{d}t < \infty .$

(A2) $\exists\, P = P^*, Q, R \in \mathcal{L}(\mathcal{U},\mathcal{U})$,

$$(\sigma(t), P\sigma(t))_U + 2(\sigma(t), Q\varphi(\sigma(t),t))_U + (\varphi(\sigma(t),t), R\varphi(\sigma(t),t))_U \leq 0 \tag{2}$$

$\forall \sigma(\cdot), \varphi(\sigma(\cdot),\cdot)\,, \sigma(\cdot)$ continuous solution of (1) , $\quad \forall t \geq 0\,,$

(Quadratic constraints)

$$L^2_\rho(\mathbb{R}_+; U) := \left\{ f \in L^2_{loc}(\mathbb{R}_+; U) : \int_0^\infty |f(t)|^2_U \, e^{2\rho t} \,\mathrm{d}t < \infty \right\}$$

is a weighted L^2-space ,

$$W^{1,2}_\rho(\mathbb{R}_+; U) := \left\{ f \in L^2_\rho(\mathbb{R}_+; U) : \dot{f} \in L^2_\rho(\mathbb{R}_+; U) \right\}$$

is a weighted Sobolev space .

(A3) The linear part of (1) is ρ-stable, i.e.

$$\exists\, \rho \geq 0 \;\forall u \in L^2(\mathbb{R}_+; U) \mapsto \sigma(\cdot) \in W^{1,2}_\rho(\mathbb{R}_+; U)\,,$$

$$\sigma(t) = \int_0^t G(t-s)u(s)\,\mathrm{d}s \quad \text{is a bounded operator.}$$

Goal: Find Hilbert spaces $Z_1 \subset Z_0 \subset Z_{-1}$ (Rigged Hilbert space structure) and linear bounded operators

$$A : Z_1 \to Z_{-1},\ B : U \to Z_{-1},\ C : Z_0 \to U$$

such that the absolute stability behaviour of (1) coincides with the absolute stability behaviour of the non-autonomous dynamical system

$$\dot{z} = Az + Bu(t), \quad \sigma(t) = Cz(t), \quad u(t) = \varphi(\sigma(t), t)\,, \tag{3}$$

and the following conditions are satisfied:

(i) $z(0, z_0, u) = 0 \quad \forall z_0 \in Z_0, \quad \forall u \in L^2_{loc}(0, \infty; U)\,,$ (Initial condition) ;

(ii) $u(t) = 0, \quad \forall t \leq T \;\Rightarrow\; z(t, 0, u) = 0, \quad \sigma(t, 0, u) = 0, \quad \forall t \leq T,$ (Causality) ;

(iii)
$$z(t + s, z_0, u) = z(t, z(s, z_0, u), \tau^s u),$$
$$\sigma(t + s, z_0, u) = \sigma(t, z(s, z_0, u), \tau^s u)$$
$$\forall z_0 \in Z_0,\ \forall u \in L^2_{loc}(0, \infty; U), \forall t, s \geq 0,$$

(Time-invariance or cocycle property)

1 Realization of a Volterra integral equation

with $\tau^s u(t) := \begin{cases} u(t+s) & \text{for} \quad t+s \geq 0\,, \\ 0 & \text{for} \quad t+s < 0 \end{cases}$ as shift semi-group.

Example 1

$$\dot{x} = d(y-x)\,, \quad \dot{y} = rx - y - xz\,, \quad \dot{z} = -bz + xy$$

$d > 0, \quad r > 0, \quad b > 0$ Lorenz equation

Nonlinearities:

$\varphi_1(x,y,z) = xz, \quad \varphi_2(x,y,z) = xy, \quad \varphi = (\varphi_1, \varphi_2)\,, \quad \sigma = (x,y,z) \in \mathbb{R}^3$

Quadratic constraints: $F(\varphi, \sigma) := \varphi_1 y - \varphi_2 z = xyz - xyz \equiv 0 \quad \hat{=}(2)$

Transfer functions: $\begin{aligned} \tilde{y} &= -\hat{G}_1(i\omega)\tilde{\varphi}_1\,, \quad \omega \in \mathbb{R} \\ \tilde{z} &= -\hat{G}_2(i\omega)\tilde{\varphi}_2 \end{aligned}$

$$\left.\begin{aligned} \hat{G}_1(p) &= \tfrac{p+d}{p^2+p(d+1)+d(1-r)} \\ \hat{G}_2(p) &= -\tfrac{1}{p+b} \end{aligned}\right\} \Rightarrow G(t)$$

Example 1 (continued)

Frequency domain condition (5) is satisfied for $0 < r < 1$
Associated Boltzmann equation in $W_\rho^{1,2}(\mathbb{R}_+; \mathbb{R}^3)$

We call this imbedding of (1) into a time-invariant control system with the same absolute stability behaviour.

Theorem 1 (Kalman [1969], Helton [1976], Salamon [1989])

Suppose that (1) is linear and the input / output process given by (1) is ρ-stable. Then there exists an imbedding of (1) into a system (3) with the same absolute stability behaviour by a Boltzmann-type transport equation, i.e. by a system (3)

$$\text{with} \quad Z_0 := W_\rho^{1,2}(\mathbb{R}_+; U),\ 0 < \rho < \rho_0\,,$$

Theorem 1 (continued)

$$Z_1 := D(A) = \{\xi : \xi(s) \in W_\rho^{1,2}(\mathbb{R}_+; U), \int\limits_0^\infty e^{2\rho s}|\ddot{\xi}(s)|^2 \mathrm{d}s < \infty\},$$

$$(A\xi)(s) := \mathrm{d}s \frac{\partial \xi(s)}{\partial s} \text{ transport or impulse operator}, \quad (4)$$

$$(B\eta)(s) := G(s)\eta,\ \eta \in U(= \mathbb{R}^n),\ Cz(s) := z(0), \forall z(s) \in W_\rho^{1,2}(\mathbb{R}_+; U).$$

Example 2

Consider

$$\dot{y} = Ay + B\varphi(\sigma(t), t), \sigma(t) = Cy(t)\,,$$

$$A - n \times n,\ B - n \times 1,\ C - 1 \times n \quad \text{matrices}\,,$$

$$\varphi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}\,,$$

$$G(t) := Ce^{At}B\,, \quad h(t) = Ce^{At}y_0\,,$$

$$\sigma(t) = h(t) + \int_0^t G(t-s)\,\varphi(\sigma(s), s)\mathrm{d}s\,.$$

Example 2 (continued)

The nonlinear Boltzmann transport equation from scattering theory

$$\mathrm{d}s\frac{\partial\sigma}{\partial t} = \mathrm{d}s\frac{\partial\sigma}{\partial x} + \int_0^x G(x-s)\varphi(\sigma(s,x))\,\mathrm{d}s$$

is a first order integro-differential equation with boundary and initial conditions

$$\sigma(t,0) = 0, \quad \sigma(0,x) = \sigma_0(x)\,.$$

Theorem 2 (Generalized Brusin's theorem; Reitmann [2011])

Consider the nonlinear Volterra integral equation (1) under the assumptions **(A1) – (A3)**. Let $\hat{G}(\lambda) := \int_0^\infty e^{-\lambda t} G(t)\,\mathrm{d}t$ be the transfer operator of the kernel. Assume that the class of nonlinearities described by **(A2)** contains

Theorem 2 (continued)

at least one linear function $\varphi(\sigma, t) = K\sigma$ with $K \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ such that the operator $(I - \hat{G}(\lambda)K)^{-1}$ has a finite number of singularities in the strip $0 < \varepsilon_1 \leq Re\,\lambda \leq \varepsilon_2$. Suppose that the frequency–domain condition

$$\hat{G}^*(i\omega)P\hat{G}(i\omega) + 2Re(Q^*\hat{G}(i\omega)) + R > 0 \quad \forall \omega \in \mathbb{R} \tag{5}$$

is satisfied. Then the nonlinear integral equation (1) can be imbedded into a non-autonomous dynamical system (3) with the same absolute stability behaviour realized with transport operator (4), i.e. there exists a linear bounded operator $M = M^* : W_\rho^{1,2}(\mathbb{R}_+; U) \to W_\rho^{1,2}(\mathbb{R}_+; U)$ with the following properties.

1) If $\sigma(t) \equiv \sigma(t, h)$ with $h \in W_\rho^{1,2}(\mathbb{R}_+; U)$ is a continuous solution of the integral equation (1) then the solution $z(t) \equiv z(t, h)$ of (3) with $z(0, h) = h$ exists and there is a positive $\delta > 0$ such that

Theorem 2 (continued)

$$\int_{t_1}^{t_2} \left(|\,\varphi(\sigma(t),t)\,|_U^2 + \|\,z(t)\,\|^2_{W_\rho^{1,2}} \right) \mathrm{d}t \le \delta\,(Mz(t),z(t))\Big|_{t_1}^{t_2} \quad \forall\, 0 \le t_1 < t_2\,.$$

2) Suppose $(\sigma(\cdot),h(\cdot))$ satisfies (1) and $(h(\cdot),Mh(\cdot)) < 0$. Then $\sigma(\cdot) \in L^2(\mathbb{R}_+;U)$, i.e. it is stable. If $(h(\cdot),Mh(\cdot)) > 0$ then $\sigma(\cdot)$ is unstable, i.e. there exists a number $\beta > 0$ such that

$$\lim_{T\to\infty} e^{-\beta T} \int_0^T |\,\varphi(\sigma(t),t)\,|_U^2\, \mathrm{d}t = \infty\,.$$

3) M is the operator solution of a <u>linear</u> integral equation.

Remark 1

The (algebraic) dimension d of the cone $\{h \in W_\rho^{1,2}(\mathbb{R}_+;U)\ :\ (Mh,h) > 0\}$ is finite and coincides with the topological dimension of an orbit closure $\mathfrak{M} := cl\{z(t), t \ge 0\}$ of system (3). Thus d real coordinates are sufficient to describe by one-to-one map the points in $\mathfrak{M}$.

Consider the nonlinear system

$$\sigma(k) = h(k) + \sum_{j=0}^{k-1} G(k-j-1)\varphi\left(\sigma(j), j\right), \tag{6}$$

$k = 1, 2, \ldots,\ \sigma(0) = \sigma_0,\ \{\sigma(k)\}_{k=1}^{\infty}$ a time-series generated by (6) ,

$h : \mathbb{N}_0 \to U\ ,\ U$: Hilbert space, $\mathbb{R}^n$,

$G(j) \in \mathcal{L}(\mathcal{U}, \mathcal{U})$,

$\varphi : U \times \mathbb{N}_0 \to U$,

$\hat{G}(p) := \sum_{k=0}^{\infty} G(k)p^{-k}$ z-transform of G ,

$T(p) := \hat{G}(1/p) = \hat{G}_0 + \hat{G}_1 p + \hat{G}_2 p^2 + \ldots$.

($\tilde{\text{A}}$1) $\exists\, c > 0 \quad \exists\, \rho_0 > 1 \;:\; \|\, G(k)\,\|_{\mathcal{L}(\mathcal{U},\mathcal{U})} \le c\rho_0^{-k}\,,\; k = 1, 2, \ldots$.[2ex]

($\tilde{\text{A}}$2) $\exists P = P^*, Q, R \in \mathcal{L}(\mathcal{U},\mathcal{U})$ s.t.
$(\sigma(k), P\sigma(k))_U + 2(\sigma(k), Q\varphi(\sigma(k), k))_U$
$+ (\varphi(\sigma(k), k),\; R\varphi(\sigma(k), k))_U \le 0$
$\forall \{\sigma(k)\}$ solution of (6), $\{\varphi(\sigma(k), k)\}\,,\; k = 1, 2, \ldots\;,$

Let $\ell^2_\rho(1, \infty; U)$ with $0 < \rho < \infty$ be the set of sequences $u = (u_1, u_2, \ldots)$ with $u_k \in U$ for which $\{\rho^{-k} u_k\}$ belongs to $\ell^2(1, \infty; U)$, i.e. $\sum_{k=1}^{\infty} \rho^{-2k}\, |u_k\,|_U^2 < \infty$.

($\tilde{\text{A}}$3) The linear part of (6) is ρ-stable, i.e. $\exists\, \rho > 0 \quad \forall u \in \ell^2(1, \infty; U)$ the sequence $\{\sigma(k)\}\,, \sigma(k) := \sum_{j=0}^{k-1} G(k - j - 1)u(j)$ belongs to $\ell^2_\rho(1, \infty; U)$.

2 Realization of a time-series

Realization procedure:

1) Introduce the backward shift $\tau : \ell^2_\rho(1,\infty;U) \to \ell^2_\rho(1,\infty;U)$ by $\tau(u_1,u_2,\dots) = (u_2,u_3,\dots),\ \forall u = (u_1,u_2,\dots) \in \ell^2_\rho(1,\infty;U)$.
   Define $A := \tau$
2) Define $B : U \to \ell^2_\rho(1,\infty;U)$ by

$$Bu := (\hat{G}_1 u, \hat{G}_2 u, \dots),\ \forall u \in U\,,$$

   and $C : \ell^2_\rho(1,\infty;U) \to U$ by

$$C(u_1,u_2,\dots) := u_1\,,\ \forall (u_1,u_2,\dots) \in \ell^2_\rho(1,\infty;U)\,.$$

3)
$$\begin{aligned} z(k+1) &= Az(k) + Bu(k)\,, \\ \sigma(k) &= Cz(k), z(0) = z_0 \in \ell^2_\rho(1,\infty;U)\,, \\ u(k) &= \varphi(\sigma(k),k)\,,\ k = 0,1,2,\dots \quad . \end{aligned} \tag{7}$$

(7) is called discrete-time Boltzmann-type transport equation associated with (6).

Example 3

$$\begin{aligned}&\sigma_{n+2}+\sigma_{n+1}+\varphi(\sigma_n,n)=0\,,\quad n=0,1,2,\dots\;,\\&\sigma(0)=\sigma_0,\quad \sigma(1)=\sigma_1\,.\end{aligned}$$

z-transform: $\quad p^2\tilde{\sigma}+p\tilde{\sigma}=-\tilde{\varphi}\,,$
$\quad \hat{G}(p)=\frac{1}{p^2+p}\,,$

$$\hat{G}\left(\frac{1}{p}\right)=\frac{1}{\frac{1}{p^2}+\frac{1}{p}}=\frac{p^2}{1+p}=p-1+\frac{1}{1+p}=p-1+\sum_{m=0}^{\infty}(-1)^m p^m$$
$$=p^2-p^3+p^4-\cdots\;.$$

$$\hat{G}_m=\begin{cases}0\,, & m=0,1\,,\\ (-1)^m\,, & m=2,3,\dots\end{cases}$$

$$\begin{aligned}&(z_1(k+1),z_2(k+2),\dots)\\&\quad=\tau(z_1(k),z_2(k),\dots)+(0,\varphi(\sigma_k,k),-\varphi(\sigma_k,k),\dots)\,,\end{aligned}$$

Example 3 (continued)

$$k = 0, 1, 2, \dots , \quad \sigma_k = z_1(k)\,,$$
$$z_m(k+1) = z_{m+1}(k) + (-1)^m\,\varphi(z_1(k), k), m, k \in \mathbb{N}_0$$

Space- and time-discrete version of the Ginsburg-Landau equation in $\ell^2_\rho(1, \infty; U)$:

$$u_j(n+1) = u_j(n) - (1 - i\beta)u_j(n)\,|\,u_j(n)\,|^2$$
$$+\ \varkappa(u_{j-1}(n) - 2u_j(n) + u_{j+1}(n))\ u_j(n) \in \mathbb{C},\ \ n, j \in \mathbb{Z}.$$

Dynamical objects of the lattice model associated to a time-series:

- finite-dimensional attractors
- hyperbolicity
- travelling waves $\quad u_j(n) = \Psi(l_j + mn)$
- spatial structures

Theorem 3

Consider the iteration (6) under the assumptions $\mathbf{(\tilde{A}1)}$ – $\mathbf{(\tilde{A}3)}$. Let $\hat{G}(p) := \sum_{k=0}^{\infty} G(k)p^{-k}$ be the z-transform of G. Assume that the class of nonlinearities described by $\mathbf{(\tilde{A}2)}$ contains at least one linear function $\varphi(\sigma, t) = K\sigma$ with $K \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ such that the operator $(I - \hat{G}(p)K)^{-1}$ has a finite number of singularities in the ring

$$1 < \varepsilon_1 \leq |\, p \,| \leq \varepsilon_2.$$

Suppose that the frequency-domain condition

$$\hat{G}^*(p) P \hat{G}^*(p) + 2Re\,(Q^* \hat{G}(p)) + R > 0, \quad \forall p \in \mathbb{C} : |\, p \,| = 1$$

is satisfied. Then there exists a linear bounded operator $M = M^* : \ell^2_\rho(1, \infty; U) \to \ell^2_\rho(1, \infty; U)$ with the following property:

Theorem 3 (continued)

Suppose $\sigma = \{\sigma(k)\}_{k=1}^{\infty}$ is a sequence generated by (6) with $h = \{h(k)\}_{k=1}^{\infty}$. Then if $(h, Mh) < 0$ we have $\sigma \in \ell^2_\rho(1, \infty; U)$, i.e. σ is stable. If $(h, Mh) > 0$ then σ is unstable.

Remark 2

$z(k) = \tau^k z(0) + \sum_{j=0}^{k-1} \tau^{k-j-1} Bu(j),\ k = 1, 2, \dots,$
$z(0) = (z_0(0), z_1(0), \dots)$ a time-series , $\mathfrak{M}(z(0)) := cl(\{\tau^k(z(0))\}_{k=0}^{\infty})$ the orbit closure. Suppose $\dim_F \mathfrak{M} =: d < \infty$ and let n be the smallest natural number s.t. $n \geq 2d + 1$.
$L := \{(z_0, z_1, \dots, z_{n-1})\}$ is an n-dimensional subspace of $\ell^2_\rho(1, \infty; U)$.
The typical projections $\ell^2_\rho(1, \infty; U) \to L$ are one-to-one. Let the standard projection π_n be typical. Then on
$E := \pi_n(\mathfrak{M}(z(0))$ there is given a dynamical system $\tilde{\tau} := \pi_n \circ \tau \circ \pi_n^{-1}$

$$(\tilde{\tau}, L) : (z_0, z_1, \dots, z_{n-1}) \mapsto (z_1, z_2, \dots, z_n).$$

We consider an ODE of the second order

$$\ddot{\sigma} + \alpha\dot{\sigma} + \varphi(\sigma(t), t) = 0 \tag{8}$$

with a smooth nonlinearity $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Assume that any solution of (6) exists on $\mathbb{R}$.
Let us rewrite (6) in the following way

$$\begin{cases} \dot{z}(t) = Az(t) + B\varphi(\sigma(t), t) \\ \sigma(t) = Cz(t)\,, \end{cases} \tag{9}$$

with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1\,, & 0 \end{pmatrix},$

where $\sigma(t)$ is the input and $\varphi(\sigma(t), t)$ is the output.
As “nonlinear part" is considered the function

$$\varphi(\sigma, t) = (\beta + \gamma \cos(t))\sigma \,, \tag{10}$$

where β and γ are parameters. Note that equation (6) with φ given by (8) has the form of the Mathieu-Hill equation.
Time is considered on the finite interval $[0, T]$.
All functions are considered as sequences

$$\{\sigma(t_i)\}_1^{N+1}, \;\; t_k = (k-1)\frac{T}{N}, k = 1, 2.., N+1$$

where $N+1$ is the number of nodes on the interval $[0, T]$.

Step 1

Find a sector for the nonlinear part such that
$\mu_1 \leq \varphi(\sigma, t)/\sigma \leq \mu_2 \quad \forall(t, \sigma) \in \mathbb{R} \times \mathbb{R}, \quad \sigma \neq 0.$
Take initial data $(\sigma_i(0), \dot{\sigma}_i(0)),\ i = 1, 2.., L$, and calculate the numbers μ_1, μ_2 such that the relation
$\mu_1 \leq \varphi(\sigma(t_i), t_i)/\sigma(t_i) \leq \mu_2\,, \quad i = 1, .., N+1\,,$
is satisfied. For the calculation of μ_1, μ_2 an adaptive algorithm is used which is finitely converging in the sense of Yakubovich [1977].

Step 2

Write system (8) as Volterra integral

$$\begin{aligned} I\sigma(t) &= h(t) + \int_0^t G(t-\tau)\varphi(\sigma(\tau),\tau)\,\mathrm{d}\tau\,, \\ \varphi(\sigma,t) &= (\beta+\gamma\cos(t))\sigma, \end{aligned} \tag{11}$$

where $h(\cdot)$ is the input and $\sigma(\cdot)$ the output ($\sigma \equiv \sigma_h$).
The goal is to construct an operator M which gives all information about stability of $\sigma_h(\cdot)$ with respect to the input $h(\cdot)$.
Assume that the kernel of (9) can be written as

$$G(t-\tau) = e^{\lambda(t-\tau)},$$

where λ is an unknown parameter.
Let $\rho \geq 0$ be the unknown parameter of the Hilbert space L^2_ρ introduced in Section 2.

Step 3

In order to construct the operator M we have as an auxiliary problem to solve the linear Fredholm integral equation of the second kind

$$\int_0^T S_{(\rho,\lambda)}(t,\tau)\tilde{u}_{h,(\rho,\lambda)}(\tau)\mathrm{d}\tau + \tilde{u}_{h,(\rho,\lambda)}(t) = g_{h,(\rho,\lambda)}(t), \qquad (12)$$

where $S_{(\rho,\lambda)}$ is a function depending on ρ and λ, and $g_{h,(\rho,\lambda)}$ depends also on $h(\cdot)$.

From this equation we get $\tilde{u}_{h,(\rho,\lambda)}(\cdot)$ which will be used further.

Remark 1

If we solve the integral equation (10) we get the solution of an associated Riccati equation. In general the Riccati equation is a quadratic equation with respect to the unknown matrix or operator. In our situation this equation (10) is linear what is important for practical realization. The reason for this is the special type of hyperbolic equations arising in (3).

Step 4

Construct the cost-functional $J^T_{\lambda,\rho}(\cdot)$ on L^2_ρ.
Take some initial values $\overline{\lambda}, \overline{\rho}$, calculate the functional with these parameters and compare with the data.
Use for this an optimization procedure with respect to λ, ρ for the functional computed along the solution of the Fredholm integral equation (10).
As result of this step we get the functional $J^T_{\lambda_0,\rho_0}$.

Step 5

Define the operator M^T by

$$(M^T h)(s) := -\mathrm{d}s \frac{1}{\lambda_1} \int_0^T \{ e^{-2\rho(\tau)} \left[e_{s-\tau} e^{\lambda_1 (s-\tau)} + e_{\tau - s} \mu_1 (s-\tau) \right] + \\ + \mu_2(-\tau) e^{\lambda_1 s} \} (P \widetilde{\sigma}_h(\tau) + Q h(\tau)) \, \mathrm{d}\tau, \ \forall h \in W^{1,2}_{\rho_0}, \quad (13)$$

where $\widetilde{\sigma}_h(t) = \int\limits_0^t (e^{\lambda_0(t-\tau)} + h(\tau))\mathrm{d}\tau + h(t)$ and the functions $\lambda_1(\cdot), \mu_1(\cdot), \mu_2(\cdot)$ depend only on ρ_0.
Then the sign of the test functional

$$< M^T h, h >= \int\limits_0^T (M^T h)(s)h(s)e^{2\rho_0 s}\,\mathrm{ds} \tag{14}$$

gives us the information about stability of $\sigma(\cdot)$ according to Theorem 2.

Consider the equation (8),(7) with the system parameters

$$\alpha = 1/3;\ \beta = 1;\ \gamma = 2. \tag{15}$$

Using the above algorithm with $T = 2\pi, N = 18, L = 50$ we find the sector from Step 1 for the "nonlinearity" (8) with $\mu_1 = -1, \mu_2 = 3$.
For the kernel $G(\cdot)$ and the function space L^2_ρ we obtain the parameters

$$\lambda_0 = 0.29,\ \rho_0 = 0.1\ . \tag{16}$$

This defines the operator M^T for the test functional (11)
In order to verify our result we consider the solution of (9) with the initial data

$$\sigma(0) = 0.15683, \dot{\sigma}(0) = 0, 25269\ . \tag{17}$$

Computing the associated h in (9) we get a positive sign of the test functional (11). According to Brusin's theorem the solution must be unstable.
The direct calculation of the solution (Fig. 3) shows their instability.
This means that the information from test functional (12) is correct.

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Thank you
for your attention!