Upper Hausdorff dimension estimates for invariant sets of evolutionary variational inequalities

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The Seventh International Conference on
Differential and Functional Differential Equations
International Workshop
"Spatio-temporal dynamical systems"

Moscow, Russia, August 22-29, 2014

Suppose that Y_0 is a real Hilbert space with $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ as scalar product resp. norm.
Suppose also that $A : \mathcal{D}(A) \subset Y_0 \to Y_0$ is an unbounded densely defined linear operator.
The Hilbert space Y_1 is defined as $\mathcal{D}(A)$ equipped with the scalar product

$$(y,\eta)_1 := ((\beta I - A)y, (\beta I - A)\eta)_0 \ , \quad y,\eta \in \mathcal{D}(A)\,, \tag{1}$$

where $\beta \in \rho(A) \cap \mathbb{R}$ ($\rho(A)$ the resolvent set of A) is an arbitrary but fixed number.
The Hilbert space Y_{-1} is by definition the completion of Y_0 with respect to the norm $\|z\|_{-1} := \|(\beta I - A)^{-1} z\|_0$. Thus we have the dense and continuous imbeddings

$$Y_1 \subset Y_0 \subset Y_{-1} \tag{2}$$

which is called *Hilbert space rigging structure*. In this triple, Y_0 is the *pivot space*, Y_1 is the *interpolation space*, and Y_{-1} is the *extrapolation space* (Triebel, 1978).

The "scalar product" $(\cdot,\cdot)_{-1,1}$ on $Y_{-1}\times Y_1$ is the unique extension by continuity of the scalar product $(\cdot,\cdot)_0$ defined on $Y_0\times Y_1$.
If $T>0$ is an arbitrary number we define the norm for Bochner measurable functions in $L^2(0,T;Y_j)$, $j=1,0,-1$, through

$$\|y(\cdot)\|_{2,j} := \left(\int_0^T \|y(t)\|_j^2\,dt\right)^{1/2}. \tag{3}$$

Let $\mathcal{W}_T$ be the space of functions $y(\cdot)\in L^2(0,T;Y_1)$ for which $\dot y(\cdot)\in L^2(0,T;Y_{-1})$ equipped with the norm

$$\|y(\cdot)\|_{\mathcal{W}_T} := (\|y(\cdot)\|_{2,1}^2 + \|\dot y(\cdot)\|_{2,-1}^2)^{1/2}. \tag{4}$$

2 Evolutionary variational inequalities

Suppose $Y_1 \subset Y_0 \subset Y_{-1}$ is a real Hilbert space rigging structure with $A \in \mathcal{L}(Y_1, Y_{-1})$. Assume that Ξ and W are two real Hilbert spaces with scalar products $(\cdot,\cdot)_\Xi$, $(\cdot,\cdot)_W$ and norms $\|\cdot\|_\Xi$, $\|\cdot\|_W$, respectively. Introduce the linear continuous operators

$$B : \Xi \to Y_{-1}\,, \quad C : Y_{-1} \to \Xi \tag{5}$$

and define the set-valued map

$$\varphi : W \to 2^{\Xi} \tag{6}$$

and the map

$$\psi : Y_1 \to \mathbb{R}_+ \cup \{+\infty\}\,. \tag{7}$$

Note that in applications φ is a *material law nonlinearity*, ψ is a *contact-type* or *friction functional* and $w(t) = Cy(t)$ is the *input* of the nonlinearity.

$$(\dot{y} - Ay - B\xi, \eta - y)_{-1,1} + \psi(\eta) - \psi(y) \geq 0\,, \quad \forall \eta \in Y_1\,, \tag{8}$$

$$w(t) = Cy(t)\,, \quad \xi(t) \in \varphi(w(t))\,, \qquad y(0) = y_0 \in Y_0\,. \tag{9}$$

2 Evolutionary variational inequalities

Remark 1

In the contact free case when $\psi \equiv 0$ the evolutionary variational inequality (8), (9) is equivalent to an *evolution equation* with a set-valued nonlinearity φ given by

$$\begin{aligned} \dot{y} &= Ay + B\xi \quad \text{in} \quad \mathrm{Y}_{-1}\,, && (8)' \\ w(t) &= Cy(t)\,, \xi(t) \in \varphi(w(t))\,, \quad y(0) = y_0 \in Y_0\,. && (9)' \end{aligned}$$

Definition 1

A function $y(\cdot) \in \mathcal{W}_T \cap C(0, T; Y_0)$, is said to be a *solution* of (8), (9) on $(0, T)$ if there exists a function $\xi(\cdot) \in L^2(0, T; \Xi)$ such that for a.a. $t \in (0, T)$ the inequality (8), (9) is satisfied and $\int_0^T \psi(y(t))dt < +\infty$. The pair $\{y(\cdot), \xi(\cdot)\}$ is called a *response* of (8), (9); $\xi(\cdot)$ is an *associated selection*.

2 Evolutionary variational inequalities

Suppose that F, G and H are quadratic forms on $Y_1 \times \Xi$. The *class* $\mathcal{N}(F,G)(\mathcal{N}(F,G,H))$ *of nonlinearities* for (8), (9) consists of all maps (6) such that the condition a) (conditions a) and b)) is (are) satisfied:

a) For any $T > 0$ and any two functions $y(\cdot) \in L^2(0,T;Y_1)$ and $\xi(\cdot) \in L^2(0,T;\Xi)$ with

$$\xi(t) \in \varphi(Cy(t)), \text{ a.a. } t \in [0,T], \tag{10}$$

it follows that

$$F(y(t),\xi(t)) \geq 0, \text{ a.a. } t \in [0,T], \tag{11}$$

and there exists a continuous function $\Phi : Y_1 \to \mathbb{R}$ (*generalized potential*) and numbers $\lambda > 0$ and $\gamma > 0$ such that

$$\int_s^t G\left(y(\tau),\xi(\tau)\right) d\tau \geq \frac{1}{2}\Big[\Phi(y(t)) - \Phi(y(s))\Big] + \lambda \int_s^t \Phi(y(\tau)d\tau \tag{12}$$

for all

$$0 \le s < t \le T$$

and

$$\Phi(y) \ge \gamma \|y\|_0^2 , \quad \forall y \in Y_0 . \tag{13}$$

b) For any $T > 0$ and any two pairs of functions

$y_1(\cdot), y_2(\cdot) \in L^2(0, T; Y_1)$ and $\xi_1(\cdot), \xi_2(\cdot) \in L^2(0, T; \Xi)$

with $\xi_i(t) \in \varphi(Cy_i(t))$, $i = 1, 2$, a.a. $t \in [0, T]$,

it follows that $H(y_1(t) - y_2(t), \xi_1(t) - \xi_2(t)) \ge 0$, a.a. $t \in [0, T]$.

(A1) For fixed linear operators A, B, C, fixed function (7), arbitrary $y_0 \in Y_0$, $T > 0$ and $\varphi \in \mathcal{N}(F, G, H)(\varphi \in \mathcal{N}(F, G))$ there exists a response $\{y(\cdot), \xi(\cdot)\}$ of (8), (9).

Example 1

Suppose that $\Omega \subset \mathbb{R}^n$ is a domain with smooth boundary $\Gamma = \partial\Omega$, $h : \Gamma \to \mathbb{R}$ is a given scalar function ("outer pressure") and $u(x,t)$ (" inner pressure") is a solution of

$$\frac{\partial u}{\partial t} = \triangle u \quad \text{in} \quad \Omega \times \mathbb{R}_+ \tag{14}$$

subject to the boundary conditions

$$u = h \quad \text{on } \Gamma \times \mathbb{R}_+ \quad \Rightarrow \quad \frac{\partial u}{\partial n} \geq 0\,, \tag{15}$$

$$u > h \quad \text{on } \Gamma \times \mathbb{R}_+ \quad \Rightarrow \quad \frac{\partial u}{\partial n} = 0 \tag{16}$$

and the initial condition

$$u(\cdot, 0) = u_0\,. \tag{17}$$

Example 1 (continued)

The system (14) – (17) describes the transfer problem of fluid acrossing a semi-permeable membrane (Lions, 1969).
Instead of (15) – (16) we consider the (nonlinear) boundary condition

$$\frac{\partial u}{\partial n} \geq g \quad \text{on} \quad \Gamma \times \mathbb{R}_+ \,, \tag{18}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given function.
In order to get a representation of (14) – (18) in the form of a variational inequality (8), (9) we introduce the spaces

$$\begin{aligned} Y_0 &:= L^2(\Omega)\,, \\ Y_1 &:= W^{1,2}(\Omega) = \{v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega)\,,\ i = 1, 2, \ldots, n\} \text{ and} \\ \Xi &:= W^{-1/2,2}(\partial\Omega)\,. \end{aligned}$$

Example 1 (continued)

An operator $A \in \mathcal{L}(Y_1, Y_{-1})$ is defined by

$$(Au, v)_{-1,1} = -\int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx\,, \quad \forall u, v \in Y_1\,. \tag{19}$$

The operator $B \in \mathcal{L}(\Xi, Y_{-1})$ is given by

$$(B\xi, y)_{-1,1} = -\int_{\partial\Omega} \xi y dS\,, \quad \forall \xi \in \Xi\,, \quad \forall y \in Y_1\,, \tag{20}$$

the nonlinear map $\varphi : Y_1 \to \Xi$ is given by

$$\varphi(y(x)) := g\,(y)(x) \quad \text{on} \quad \Gamma\,, \tag{21}$$

and the "contact functional" $\psi : Y_1 \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\psi(\eta) := \begin{cases} 0 & \text{if } \eta(x) \geq h(x) \text{ on } \Gamma\,, \\ +\infty & \text{in other cases.} \end{cases} \tag{22}$$

Example 1 (continued)

Thus the transfer problem of fluid (14) – (18) can be considered as evolutionary variational inequality

$$(\dot{y} - Ay - B\xi\,,\,\eta - y)_{-1,1} + \psi(\eta) - \psi(y) \geq 0\,,\quad \forall \eta \in Y_1\,, \tag{23}$$

$$\xi(t) = \varphi(y(t))\,,\quad y(0) = y_0 \in Y_0\,. \tag{24}$$

Let us describe the class $\mathcal{N}(F, G)$ for (23), (24). We assume that the nonlinearity φ from (21) has the following two properties:

(H1) $\exists \mu_0 > 0 \quad \forall y_1, y_2 \in Y_1 \quad :$

$$0 \leq (B\varphi(y_1) - B\varphi(y_2)\,,\,y_1 - y_2)_{-1,1} \leq \mu_0 \|y_1 - y_2\|_1^2\,. \tag{25}$$

(H2) There exist a Fréchet differentiable map $\Phi : Y_0 \to \mathbb{R}$ and a number $\lambda > 0$ such that with the Fréchet derivative $\Phi' \in \mathcal{L}(Y_0, \mathbb{R})$ the inequality

$$(\varphi(y), \eta)_1 \geq \Phi'(y)\eta + \lambda\Phi(\eta)\,,\quad \forall \eta \in Y_1 \qquad \text{is satisfied.} \tag{26}$$

Example 1 (continued)

It is clear that (25) and (26) can be considered as a monotonicity condition and a potential-type condition, respectively. Using (25) we can introduce the quadratic form

$$F(y,\xi) := \mu_0 \|y\|_1^2 - (B\xi, y)_{-1,1}\,, \quad (y,\xi) \in Y_1 \times \Xi\,, \tag{27}$$

which satisfies (11). The inequality (26) can be used to define the quadratic form

$$G(y,\xi) := (G_1 A y, \xi)_\Xi + (G_2 B\xi, \xi)_\Xi \quad \text{on} \quad Y_1 \times \Xi \tag{28}$$

with $G_i : Y_{-1} \to \Xi \quad (i = 1, 2)$. It is easy to see that the form G from (28) and the generalized potential Φ from (26) satisfy the inequality (12) .

a) *Observations that are determining for the dissipativity*

Suppose S is a real Hilbert space (*observation space*), $M : Y_1 \to S$ is a given linear bounded operator (*observation operator*), $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1), P = P^*$ in Y_0, is also given such that the following conditions are satisfied.

1) $V_1(y) := \frac{1}{2}(y, Py)_0 \geq 0 \quad , \quad \forall y \in Y_0$;
2) $V(y) := V_1(y) + \frac{1}{2}\Phi(y) \geq \mathrm{const} \cdot \|y\|_0^2 , \quad \forall\, y \in Y_0$;
3) There exist numbers $\lambda > 0$ and $\mu > 0$ such that for an arbitrary solution $y(\cdot)$ of (8), (9) the function $m(t) := V(y(t))$ satisfies

$$\dot{m}(t)+2\,\lambda m(t)+\psi(y\,(t))-\psi(-Py(t)+y(t)) \leq \mu \|My(t)\|_S^2 \,, \ \text{a.a. } t \geq 0. \tag{29}$$

Then the observation

$$\sigma(t) := \mu \|My(t)\|_S^2 \tag{30}$$

is *determining for the dissipativity with domain* $\mathcal{D}$ of (8), (9), i.e., the property

$$\int_t^{t+1} \|My(\tau)\|_S^2 \, d\tau \to 0 \quad \text{for} \quad t \to +\infty$$

implies that

$$\limsup_{t \to +\infty} m(t) \leq C \quad \text{and,}$$

consequently, (8), (9) is (point) *dissipative* with *domain of dissipativity*

$$\mathcal{D} := \left\{ y \in Y_0 : \|y\|_0^2 \leq \frac{2C}{\gamma} \right\}. \tag{31}$$

b) *Observations that are determining for the complete deviation of arbitrary two solutions*

Suppose $M \in \mathcal{L}(Y_1, S)$ is given as in a). Suppose also that there exist an operator $P_1 \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$, $P_1 = P_1^*$ in Y_0 , numbers $\lambda_1 > 0, \alpha_1 > 0, \delta_1 > 0$ and $\mu_1 > 0$ such that for arbitrary two solutions $y_1(\cdot), y_2(\cdot)$ of (8), (9) the function

$$m_1(t) := \Big(y_1(t) - y_2(t),\, P_1\big(y_1(t) - y_2(t)\big)\Big)_0$$

satisfies for a.a. $t > 0$ the inequality

$$\begin{aligned} \dot{m}_1(t) + 2\,\lambda_1 m_1(t) + \psi(y_1\,(t)) - \psi(y_1(t) - P_1(y_2(t) - y_1(t))) \\ - \psi(y_2(t) + P_1(y_1(t) - y_2(t))) + \psi(y_2(t)) \\ + \delta_1 \| e^{-\alpha_1 t}(y_1(t) - y_2(t))\|_0^2 \le \mu_1 \| M(y_1(t) - y_2(t))\|_S^2 \,. \end{aligned} \tag{32}$$

Then the observation $\sigma_1(t) = \mu_1 \| M(y_1(t) - y_2(t))\|_S^2$ is *determining for the complete deviation* $y_1(t) - y_2(t)$, i.e. , the property

$$\int_t^{t+1} \|M(y_1(\tau) - y_2(\tau))\|_S^2 \, d\tau \to 0 \quad \text{for} \quad t \to +\infty \tag{33}$$

implies that for a.a. $t > 0$

$$\|y_1(t) - y_2(t)\|_0^2 \le c_1 e^{2\alpha_1 t} \|y_1(0) - y_2(0)\|_0^2 \, , \tag{34}$$

where $c_1 > 0$ is a certain constant not depending on the solutions. The inequality (34) follows from (32) since

$$\int_0^{\infty} \|e^{-\alpha_1 t}(y_1(t) - y_2(t))\|_S^2 \, dt < +\infty. \tag{35}$$

c) *Observations that are determining for the convergence in a subspace of codimension n*

Suppose $M \in \mathcal{L}(Y_1, S)$ is given as in a). Suppose also that there exist an operator
$P_2 \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1), P_2 = P_2^*$ in Y_0, a natural number n, numbers $\lambda_2 > 0$,
$\alpha_2 > 0$, $\delta_2 > 0$ and $\mu_2 > 0$ such that for arbitrary two solutions $y_1(\cdot), y_2(\cdot)$ of (8), (9) the function

$$m_2(t) := (y_1(t)) - y_2(t)\, , P_2(y_1(t) - y_2(t)))_0$$

satisfies for a.a. $t > 0$ the inequality

$$\begin{aligned} \dot{m}_2(t) + 2\,\lambda_2 m_2(t) + \psi(y_1\,(t)) - \psi(y_1(t) - P_2(y_2(t) - \,y_1(t))) \\ -\psi(y_2(t) + P_2(y_1(t) - y_2(t))) + \,\psi(y_2(t)) \\ +\delta_2 \| e^{-\alpha_2 t}(1 - \pi_n)(y_1(t) - y_2(t))\|_0^2 \le \mu_2 \| M(y_1(t) - y_2(t)\|_S^2\, . \quad (36) \end{aligned}$$

Then the observation $\sigma_2(t) := \mu_2 \|M(y_1(t) - y_2(t))\|_S^2$ is *determining for the convergence in a subspace of* Y_1 *of codimension* n, i.e., the property

$$\int_t^{t+1} \|M(y_1(\tau) - y_2(\tau))\|_S^2 \, d\tau \to 0 \quad \text{for} \quad t \to +\infty \tag{37}$$

implies that for a.a. $t > 0$

$$\|(1 - \pi_n)(y_1(t) - y_2(t))\|_0^2 \leq c_2 e^{-2\alpha_2 t} \|y_1(0) - y_2(0)\|_0^2 \,, \tag{38}$$

where $c_2 > 0$ is a certain constant not depending on the solutions. Again the inequality (38) follows from (36) since

$$\int_0^{\infty} \|e^{\alpha_2 t}(y_1(t) - y_2(t))\|_0^2 \, dt < +\infty \,. \tag{39}$$

Remark 2

Determining observations (also called "determining functionals") are introduced by Foias and Prodi, 1967, Ladyzhenskaya, 1975, Foias and Temam, 1984, Chueshov, 1996, 1999. Inverse problems for variational inequalities (parameter identification problems) are considered by Hoffmann and Sprekels, 1986, Maksimov, 1992 and other authors.

Theorem 1

Suppose that for the variational inequality (8), (9) there exist observations that are determining for the dissipativity with domain $\mathcal{D}$, determining for the complete deviation and determining for the convergence in a subspace of codimension n, respectively. Then any positively invariant for (8), (9) compact set in $\mathcal{D}$ has a finite fractal dimension.

Idea of proof: The inequalities (29), (34) and (38) are the essential sufficient parts for the use of Ladyzhenskaya's theorem (see also Chuesov's version of this theorem in Chueshov, 1999).

Theorem 2 (Ladyzhenskaya, 1975)

Suppose $\mathcal{K}$ is a compact set in the Hilbert space $(Y, \|\cdot\|)$ and $\phi : \mathcal{K} \to \phi(\mathcal{K})$ is a continuous map with $\mathcal{K} \subset \phi(\mathcal{K})$ and such that

$$\|\phi(y) - \phi(\eta)\| \leq l \|y - \eta\|, \quad \|(1 - \pi_n)(\phi(y) - \phi(\eta))\| \leq q \|y - \eta\|, \ \forall y, \eta \in Y.$$

Here $l \geq 0$, $0 \leq q < 1$ are constants, π_n is the orthoprojector in Y on a subspace of dimension n.

Then $\quad \dim_F \mathcal{K} \leq n \, \ln \dfrac{2\nu^2 l^2}{1 - q^2} \left(\ln \dfrac{2}{1 + q^2} \right)^{-1}$ (ν is an absolute constant).

(A2) There exists a number $\lambda > 0$ such that for any $T > 0$ and any $f \in L^2(0, T; Y_{-1})$ the problem

$$\dot{y} = (A + \lambda I)y + f(t)\ ,\ \ y(0) = y_0 \tag{40}$$

is *well-posed*, i.e., for arbitrary $y_0 \in Y_0, f(\cdot) \in L^2(0, T; Y_{-1})$ there exists an unique solution $y(\cdot) \in \mathcal{W}_T$ satisfying (40) in the sense

$$(\dot{y}, \eta)_{-1,1} = ((A + \lambda I)y, \eta)_{-1,1} + (f(t)\ , \eta)_{-1,1}\ , \quad \forall \eta \in Y_1,\ \text{a.a. } t \in [0, T]\ , \tag{41}$$

and depending continuously on the initial data, i.e.,

$$\|y(\cdot)\|^2_{\mathcal{W}_T} \le c_1 \|y_0\|^2_0 + c_2 \|f(\cdot)\|^2_{2,-1}\ , \tag{42}$$

where $c_1 > 0$ and $c_2 > 0$ are some constants. Furthermore, any solution of

$$\dot{y} = (A + \lambda I)y, \quad y(0) = y_0 \tag{43}$$

is exponentially decreasing for $t \to +\infty$, i.e., there exist constants $c_3 > 0$ and $\varepsilon > 0$ such that

$$\|y(t)\|_0 \le c_3 e^{-\varepsilon t} \|y_0\|_0\ ,\ \ t > 0\ . \tag{44}$$

(A3) There exists a number $\lambda > 0$ such that the operator $A + \lambda I \in \mathcal{L}(Y_1, Y_{-1})$ is *regular*, i.e., for any $T > 0, y_0 \in Y_1, z_T \in Y_1$ and $f \in L^2(0, T; Y_0)$ the solutions of the *direct problem*

$$\dot{y} = (A + \lambda I)y + f(t), \quad y(0) = y_0, \quad \text{a.a. } t \in [0, T], \tag{45}$$

and of the *dual problem*

$$\dot{z} = -(A + \lambda I)^* z + f(t), \quad z(0) = z_T, \quad \text{a.a. } t \in [0, T], \tag{46}$$

are strongly continuous in t in the norm of Y_1.

The elements of the complexification Y_0^c of the real Hilbert space Y_0 can be written as $x + iy$ with $x, y \in Y_0$, and the inner product of Y_0^c will be denoted by $(\cdot, \cdot)_{Y_0^c}$. The complexification of the other spaces are defined in a similar way.

4 Frequency-domain conditions for the existence of determining observations

For the linear operator $A : Y_1 \to Y_{-1}$ we denote by A^c the linear operator $A^c : Y_1^c \to Y_{-1}^c$ defined by $A^c(x + iy) = Ax + iAy$. Again, the complexification of the other linear operators which will appear below, is defined in a similar way. Consider now the complexification of the quadratic form F (similarly of G). Suppose that

$$F(y, \xi) = (F_1 y, y)_{-1,1} + 2(F_2 y, \xi)_\Xi + (F_3 \xi, \xi)_\Xi \tag{47}$$

for $(y, \xi) \in Y_1 \times \Xi$, where $F_1 = F_1^* \in \mathcal{L}(Y_1, Y_{-1}), F_2 \in \mathcal{L}(Y_1, \Xi)$ and $F_3 = F_3^* \in \mathcal{L}(\Xi, \Xi)$.

The complexification of the quadratic form (47) is the Hermitian form F^c defined on $Y_1^c \times \Xi^c$ by

$$F^c(y, \xi) = (F_1^c y, y)_{Y_{-1}^c, Y_1^c} + 2\mathrm{Re}\,(F_2^c y, \xi)_{\Xi^c} + (F_3^c \xi, \xi)_{\Xi^c}\,. \tag{48}$$

(A4) (Frequency-domain condition)
There exist numbers $\lambda > 0$ and $\mu > 0$ such that the following two properties hold:

$$\text{a) } F^c(y,\xi) + G^c(y,\xi) - \mu\|M^c y\|^2_{S^c} \leq 0 \tag{49}$$
$$\forall\, (y,\xi) \in Y_1^c \times \Xi^c : \exists\, \omega \in \mathbb{R} \quad \text{with} \quad i\omega\, y = (A^c + \lambda I^c)y + B^c\,\xi\,;$$
$$\text{b) The functional} \qquad J(y(\cdot),\xi(\cdot)) := \tag{50}$$
$$\int_0^\infty [F^c(y(\tau),\xi(\tau)) + G^c(y(\tau),\xi(\tau)) - \mu\|M^c y(\tau)\|^2_{S^c}]\, d\tau$$

is bounded from above on the set

$$\mathcal{M}_{y_0} := \Big\{ y(\cdot),\xi(\cdot) : \dot{y} = (A^c + \lambda I^c)y + B^c\xi,$$
$$y(0) = y_0\,,\ y(\cdot) \in \mathcal{W}^c_\infty\,,\ \xi(\cdot) \in L^2(0,\infty;\Xi^c) \Big\}$$

for any $y_0 \in Y_0^c$, i.e., for any such y_0 there exists a $\gamma(y_0) \in \mathbb{R}$ such that $J(y(\cdot),\xi(\cdot)) \leq \gamma(y_0)$.

Theorem 3

Suppose that there exist numbers $\lambda > 0$ and $\delta > 0$ such that the assumptions **(A1)** - **(A4)** are satisfied for (6) - (9) with $\varphi \in \mathcal{N}(F, G)$ and an observation given by (30). Then the observation (30) is determining for the dissipativity of (8), (9) with domain $\mathcal{D}$ given by (31).

Idea of the proof: We try to find an operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ with $(y, Py)_0 \geq 0\,, \quad \forall y \in Y_0$, and numbers $\lambda > 0, \mu > 0$ such that for any solution $y(\cdot)$ of (8), (9) and their associated generalized potential Φ from condition (12) the integrated inequality (29) is true on any time interval $0 < s < t$, i.e.,

$$m(t) - m(s) + 2\lambda \int_s^t m(\tau)d\tau + \int_s^t p(\tau)d\tau \leq \int_s^t g(\tau)d\tau\,. \tag{51}$$

4 Frequency-domain conditions for the existence of determining observations

In (51) we have introduced the functions

$$m(t) := \frac{1}{2}\big(y(t), Py(t)\big)_0 + \frac{1}{2}\Phi(y(t))\,, \tag{52}$$

$$p(t) := \psi(y(t)) - \psi\big(y(t) - Py(t)\big)\,, \tag{53}$$

and

$$g(t) := -\mu\|My(t)\|_S^2\,. \tag{54}$$

In order to guarantee the inequality (51) we choose an operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and numbers $\lambda > 0, \mu > 0$ such that

$$(-(A+\lambda I)v - B\zeta, Pv)_{-1,1} \geq F(v,\zeta) + G(v,\zeta) - \mu\|Mv\|_S^2\,,\ \forall y \in Y_1\,,\ \forall \zeta \in \Xi\,. \tag{55}$$

4 Frequency-domain conditions for the existence of determining observations

The existence of such a P with $(y, Py)_0 \geq 0\,, \quad \forall y \in Y_0\,,$ follows due to the assumptions **(A2)** $-$ **(A4)** from the infinite-dimensional version of the Kalman-Yakubovich-Popov Lemma (Frequency Theorem (Brusin, 1976, Likhtarnikov, Yakubovich, 1976). From (8), (9) it follows with $v := y(t)$ and $\zeta := \xi(t)$ that

$$(\dot{y}(t), Py(t))_{-1,1} + \lambda(y(t), Py(t))_0 - ((A + \lambda I)y(t) + B\xi(t), Py(t))_{-1,1} + p(t) \leq 0\,, \qquad \text{a.a. } t > 0\,. \tag{56}$$

Using the estimate (55) we derive from (56) the inequality

$$(\dot{y}(t), Py(t))_{-1,1} + \lambda(y(t), Py(t))_0 + F(y(t), \xi(t)) + G\,(y(t), \xi(t)) - \mu\|My(t)\|_S^2 + p(t) \leq 0\,, \quad \text{a.a. } t > 0\,. \tag{57}$$

Integration of (57) on the time interval $0 < s < t$ gives

$$\frac{1}{2}(y(t), Py(t))_0 - \frac{1}{2}(y(s), Py(s))_0 + \lambda \int_s^t (y(\tau), Py(\tau))_0 \, d\tau$$
$$+ \int_s^t F(y(\tau), \xi(\tau)) d\tau + \int_s^t G(y(\tau), \xi(\tau)) d\tau + \int_s^t p(\tau) d\tau$$
$$\leq \mu \int_s^t \|Mv(t)\|_S^2 \, d\tau \, . \tag{58}$$

From the inequalities (11) and (12) it follows that

$$\int_s^t F(y(\tau), \xi(\tau)) d\tau \geq 0 \tag{59}$$

and

$$\int_s^t G(y(\tau), \xi(\tau)) d\tau \geq \frac{1}{2}\Big[\Phi\big(y(t)\big) - \Phi\big(y(s)\big)\Big] + \lambda \int_s^t \Phi(y(\tau)) d\tau \, ,$$
$$0 < s < t \, . \tag{60}$$

4 Frequency-domain conditions for the existence of determining observations

Taking into account now (58) − (60) we obtain that

$$\frac{1}{2}(y(t), Py(t))_0 + \frac{1}{2}\Phi(y(t)) - \frac{1}{2}(y(s), Py(s))_0 - \frac{1}{2}\Phi(y(s)) \tag{61}$$
$$+2\lambda \int_s^t \left[\frac{1}{2}\big(y(\tau), Py(\tau)\big)_0 - \frac{1}{2}\Phi\big(y(\tau)\big)\right] d\tau + \int_s^t p(\tau) d\tau$$
$$\leq \mu \int_s^t \|My(\tau)\|_S^2 \, d\tau \, .$$

Now, we conclude that (61) implies the inequality (51) with the functions $m(\cdot), p(\cdot)$ and $g(\cdot)$ defined by (52) − (54) .

4 Frequency-domain conditions for the existence of determining observations

Remark 2

The frequency-domain condition **(A4)** depends on imbedding properties of the Sobolev spaces under consideration. Assume, for example, that $G \equiv 0$ and

$$F(y,\xi) = \beta_0 \|y\|_0^2 - \beta_1 \|y\|_1^2\,, \quad (y,\xi) \in Y_0 \times \Xi\,, \tag{62}$$

where β_0 and β_1 are certain real constants.
In order to verify (49) we introduce the frequency-domain characteristic

$$\chi(i\omega) := (i\omega I^c - A_\lambda^c)^{-1} B^c \tag{63}$$

for $\omega \in \mathbb{R}$ s. t. $i\omega \in \rho(A_\lambda^c)$, where $A_\lambda^c := A^c + \lambda I^c$.

Remark 2 (continued)

It follows that the frequency-domain condition (49) is satisfied if

$$\beta_0 \|\chi(i\omega)\xi\|^2_{Y_0^c} - \beta_1 \|\chi(i\omega)\xi\|^2_{Y_1^c} - \delta \|M^c \chi(i\omega)\xi\|^2_{S^c} \le 0\,, \qquad \forall\, \xi \in \Xi^c\,, \quad \forall\, \omega \in \mathbb{R}\ : \quad i\omega \in \rho(A^c_\lambda)\,. \tag{64}$$

Suppose now that from the imbedding $Y_1^c \subset Y_0^c \subset Y_{-1}^c$ and the properties of the observation operator M we have the a priori estimate

$$\|v\|^2_{Y_0^c} \le c_1 \|v\|^2_{Y_1^c} + c_2 \varepsilon_{M^c} \|M^c v\|^2_{S^c}\,, \quad \forall v \in Y_1^c\,, \tag{65}$$

where $c_1 > 0$ and $c_2 > 0$ are certain constants and

$$\varepsilon_{M^c} = \varepsilon_{M^c}(Y_1^c, Y_0^c) := \sup\{\|w\|_{Y_0^c} : w \in Y_1^c\,,\ M^c w = 0_{S^c}\,, \|w\|_{Y_1^c} \le 1\} \tag{66}$$

4 Frequency-domain conditions for the existence of determining observations

Remark 2 (continued)

is the *completeness defect* of the observation operator M^c with respect to the imbedding $Y_1^c \subset Y_0^c$.

It follows from (65) that the frequency-domain condition (64) is satisfied if

$$\beta_0\, c_1 \|\chi(i\omega)\xi\|^2_{Y_1^c} - \beta_1 \|\chi(i\omega)\xi\|^2_{Y_1^c} + \beta_0 c_2 \varepsilon_{M^c} \|M^c \chi(i\omega)\xi\|^2_{S^c} - \mu \|M^c \chi(i\omega)\xi\|^2_{S^c} \le 0 \qquad \forall\, \xi \in \Xi^c\,, \quad \forall \omega \in \mathbb{R} : \quad i\omega \in \rho(A^c_\lambda)\,. \tag{67}$$

For (67) it is sufficient that

$$\beta_0\, c_1 - \beta_1 \le 0 \quad \text{and} \quad \beta_0\, c_2\, \varepsilon_{M^c} - \delta \le 0\,. \tag{68}$$

4 Frequency-domain conditions for the existence of determining observations

Remark 2 (continued)

We see that if $\beta_0 c_1 - \beta_1 \leq 0$ the second condition of (68) is always satisfied if the completeness defect of the observation operator is small. In this case, assuming that the other assumptions for the Theorem 5.1 are also satisfied, it follows that the observation $\sigma(t) = My(t)$ is determining for the dissipativity .

Suppose that $M_k y := (l_1(y), \ldots, l_k(y))$, where $l_i : Y_1 \to \mathbb{R}\,,\ i = 1, \ldots, k$, are continuous linear functionals and $Y_1 = W^{s,2}(\Omega)$, $Y_0 = W^{\sigma,2}(\Omega)$ with $s > \sigma$. Then $\varepsilon_{M^c} \approx c_1(\frac{c_2}{k})^{s-\sigma}$, i.e., the completeness defect of the observation operator M_k depends on the smoothness properties of the imbedding $Y_1^c \subset Y_0^c$ (Triebel, 1978).

5 Determining observations for second-order visco-elastic contact problems

A typical frictional contact problem is modeled by the following second-order evolutionary variational inequality (Duvant, Lions, 1976, Han, Sofonea, 2000, Jarucek, Eck, 1996): Find a displacement function u such that for a.a. $t \in [0, T]$

$$
\begin{aligned}
(\ddot{u}(t), v - \dot{u}(t))_{\mathcal{V}_{-1},\mathcal{V}_1} + (\mathcal{A}\,\dot{u}(t), v - \dot{u}(t))_{\mathcal{V}_{-1},\mathcal{V}_1} \\
+ \Big(g\big(u(t)\big), v - \dot{u}(t)\Big)_{\mathcal{V}_{-1},\mathcal{V}_1} + j(v) - j\big(\dot{u}(t)\big) \geq 0\,, \quad \forall\, v \in \mathcal{V}_1\,,
\end{aligned}
\tag{69}
$$

$$
u(0) = u_0 \in \mathcal{V}_1\,,\ \dot{u}(0) = v_0 \in \mathcal{V}_0\,. \tag{70}
$$

Here $\mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1}$ is a Hilbert space rigging structure, $\mathcal{A} : \mathcal{V}_1 \to \mathcal{V}_{-1}$ is a linear continuous operator which is called *viscosity operator*.
The nonlinear map $g : \mathcal{V}_1 \to \mathcal{V}_{-1}$ is the *elasticity operator* and $j : \mathcal{V}_1 \to \mathbb{R}_+$ represents the *contact functional*.

5 Determining observations for second-order visco-elastic contact problems

Under a solution u of (69), (70) on $(0, T)$ we understand a function $u(\cdot) \in L^2(0, T; \mathcal{V}_1)$ such that $\dot{u}(\cdot) \in L^2(0, T; \mathcal{V}_1), \ddot{u}(\cdot) \in L^2(0, T; \mathcal{V}_1$, $\int_0^T j(\dot{u}(\tau))d\tau < \infty$, and (69), (70) is satisfied for a.a. $t \in (0, T)$.

Let us assume that for any $(u_0, v_0) \in \mathcal{V}_1 \times \mathcal{V}_0$ and any time $T > 0$ a solution of (69), (70) exists. In order to rewrite (69), (70) as a first-order variational inequality (8), (9) we define the product Hilbert space rigging structure $Y_1 \subset Y_0 \subset Y_{-1}$ with

$$Y_0 = \mathcal{V}_1 \times \mathcal{V}_0\,, \quad Y_1 = \mathcal{V}_1 \times \mathcal{V}_1\,, \quad Y_{-1} = \mathcal{V}_0 \times \mathcal{V}_{-1}\,. \tag{71}$$

Let us introduce the new variables $y_1 = u$, $y_2 = \dot{u}$ and $\eta_2 = v$. It follows that $\dot{y}_1 = y_2$ and $\dot{y}_2 = \ddot{u}$. In this notation the variational inequality (69) can be rewritten as

$$(\dot{y}_2, \eta_2 - y_2)_{\mathcal{V}_{-1},\mathcal{V}_1} + (\mathcal{A}y_2, \eta_2 - y_2)_{\mathcal{V}_{-1},\mathcal{V}_1} + \big(g(y_1), \eta_2 - y_2\big)_{\mathcal{V}_{-1},\mathcal{V}_1}$$
$$+j(\eta_2) - j(y_2) \ge 0\,, \quad \forall\, \eta_2 \in \mathcal{V}_1\,, \tag{72}$$

5 Determining observations for second-order visco-elastic contact problems

Using the product topology we get for arbitrary $y = (y_1, y_2) \in Y_{-1} = \mathcal{V}_0 \times \mathcal{V}_{-1}$ and $\eta = (\eta_1, \eta_2) \in Y_1 = \mathcal{V}_1 \times \mathcal{V}_1$ the representation of the duality pairing on $Y_{-1} \times Y_1$ as

$$(y, \eta)_{-1,1} = (y_1, \eta_1)_{\mathcal{V}_1} + (y_2, \eta_2)_{\mathcal{V}_{-1}, \mathcal{V}_1} \,. \tag{73}$$

It follows from (73) that

$$(\dot{y}_2, \eta_2 - y_2)_{\mathcal{V}_{-1}, \mathcal{V}_1} = (\dot{y}, \eta - y)_{-1,1} - (y_2, \eta_1 - y_1)_{\mathcal{V}_1} \,. \tag{74}$$

A linear bounded operator $A : Y_1 \to Y_{-1}$ is defined by

$$\begin{aligned} &(-Ay, \eta - y)_{-1,1} = -(y_2, \eta_1 - y_1)_{\mathcal{V}_1} + (\mathcal{A}y_2, \eta_2 - y_2)_{\mathcal{V}_{-1}, \mathcal{V}_1} \,, \\ &\forall\; y = (y_1, y_2), \eta = (\eta_1, \eta_2) \in Y_1 = \mathcal{V}_1 \times \mathcal{V}_1 \,. \end{aligned} \tag{75}$$

It is easy to see that A defined by (75) has the representation

$$A = \begin{bmatrix} 0 & I \\ 0 & -\mathcal{A} \end{bmatrix} . \tag{76}$$

5 Determining observations for second-order visco-elastic contact problems

In order to determine the linear operator $B : \Xi = \mathcal{V}_1 \to Y_{-1}$ we use the equation

$$(-B\varphi(y_1), \eta - y)_{-1,1} = (\varphi(y_1), \eta_2 - y_2)_{\mathcal{V}_{-1},\mathcal{V}_1}, \\ \forall\, y = (y_1, y_2), \eta = (\eta_1, \eta_2) \in Y_1 = \mathcal{V}_1 \times \mathcal{V}_1 \,. \tag{77}$$

From (77) it follows that

$$B\varphi(Cy) = \begin{bmatrix} 0 \\ -\varphi(y_1) \end{bmatrix}, \tag{78}$$

where the linear operator $C : Y_1 \to W := \mathcal{V}_1$ is defined by $(y_1, y_2) \mapsto y_1$.

The last remainig element in the inequality (8) is the contact functional $\psi : Y_1 \to \mathbb{R}_+$ given by

$$\psi(y) := j(y_2)\,, \quad \forall\, (y_1, y_2) \in Y_1 = \mathcal{V}_1 \times \mathcal{V}_1 \,. \tag{79}$$

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Thank you for your attention!