Bifurcation on a finite time interval in nonlinear hyperbolic-parabolic parameter dependent control systems

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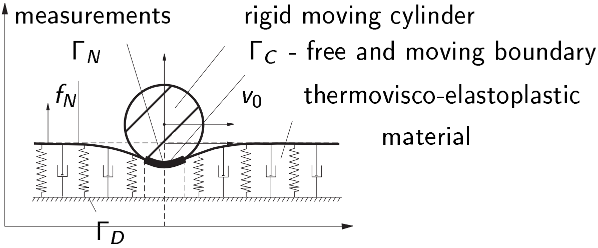
1.1 The mechanical model



1.2 Notation

Suppose $\Omega \subset \mathbb{R}^m$ is a domain, $\Gamma = \partial\Omega$ is the piecewise Lipschitz continuous boundary divided into the three disjunct parts Γ_D, Γ_N and Γ_C. Assume that $x = (x^1, \dots, x^m)$ is the location in $\Omega, t \in \mathbb{R}_+$ is the time, $n = (n^1, \dots, n^m)$ is the unit normal to Γ, $u(x,t) = (u^1(x,t), \dots, u^m(x,t))$ are the displacements, $\Theta = \Theta(x,t)$ is the temperature, $\sigma = (\sigma^{ij})$ is the stress tensor, $f_A = (f_A^1(x,t), \dots, f_A^m(x,t))$ are the body forces in Ω and $\kappa = \kappa(x,t)$ is the density of heat sources.

1.3 Elastoplastic and heat equations

The *equations of motion* and *heat transfer* are given by

$$[\sigma^{kj}(\delta^i_k + u^i_{,k})]_{,j} + f^i_A = \ddot{u}^i \text{ in } \Omega \times (0, T)\,, \tag{1}$$

$$\dot{\Theta} - (k^{ij}\Theta_{,j})_{,i} = -c^{ij}u_{i,j} + \kappa \text{ in } \Omega \times (0, T)\,, \tag{2}$$

where $c^{ij} = c^{ij}(x)$ and $k^{ij} = k^{ij}(x)$ are the tensors of thermal expansion and thermal conductivity, respectively, and σ is defined by the *thermovisco-elastoplastic stress-strain relation*

$$\sigma^{ij} = a^{ijkl}u_{k,l} + b^{ijkl}\dot{u}_{k,l} - c^{ij}\Theta + \mathcal{P}^{ij}[u_{k,l}, \Theta] \text{ in } \Omega \times (0, T)\,, \tag{3}$$

where (a^{ijkl}) and (b^{ijkl}) are the tensors of elastic and viscosity coefficients, respectively, $\{\mathcal{P}^{ij}[\cdot, \Theta]\}_{\Theta > 0}$ is the plastic part given by Θ-dependent hysteresis operators.

As *boundary and initial conditions* we have:

a) *Prescribed displacements and temperature*

$$\begin{aligned} &u = 0 \quad \text{on} \quad \Gamma_D \times (0, T)\,; \\ &\Theta = \Theta_b \quad \text{on} \quad (\Gamma_D \cup \Gamma_N) \times (0, T)\,; \\ &u(\cdot, 0) = u_0\,,\ \dot{u}(\cdot, 0) = u_1, \Theta(\cdot, 0) = \Theta_0 \text{ in } \Omega\,; \end{aligned} \tag{4}$$

b) *Prescribed boundary forces*

$$\sigma^{ij} n_j = f_N^i \quad \text{on} \quad \Gamma_N \times (0, T)\,, \tag{5}$$

where $f_N = (f_N^i(x, t))$ are the applied tractions;

c) *Frictional stress and temperature on Γ_C*

By Coulomb's law of dry friction

$$\begin{aligned} &|\sigma_{\mathcal{T}}| \le \mu|\sigma_{\mathcal{N}}|(1 - \delta|\sigma_{\mathcal{N}}|)_+ \text{ on } \Gamma_C \times (0, T)\,, \\ &|\sigma_{\mathcal{T}}| < \mu|\sigma_{\mathcal{N}}|(1 - \delta|\sigma_{\mathcal{N}}|)_+ \;\Rightarrow\; \dot{u}_{\mathcal{T}} = v_0 \quad \text{(stick zone)}\,, \\ &|\sigma_{\mathcal{T}}| = \mu|\sigma_{\mathcal{N}}|(1 - \delta|\sigma_{\mathcal{N}}|)_+ \;\Rightarrow\; \dot{u}_{\mathcal{T}} = v_0 - \lambda\sigma_{\mathcal{T}} \quad \text{(slip zone)}\,, \end{aligned} \tag{6}$$

$$k^{ij}\Theta_{,i}n_j = \mu|\sigma_{\mathcal{N}}|(1-\delta|\sigma_{\mathcal{N}}|)_+ s_C(\cdot,|\dot{u}_{\mathcal{T}}-v_0|) - k_e(\Theta-\Theta_R)\,, \tag{7}$$

where $\sigma_{\mathcal{N}} = \sigma^{ij}n_in_j$ and $u_{\mathcal{N}} = u^in_i$ are the normal components of σ and u on Γ, respectively, $\sigma^i_{\mathcal{T}} = \sigma^{ij}n_j - \sigma_{\mathcal{N}}n^i$ and $u^i_{\mathcal{T}} = u^i - u_{\mathcal{N}}n^i$ are the tangential components of σ and u on Γ, respectively, μ is the friction coefficient, v_0 is the velocity of the moving rigid body, δ is a positive constant, Θ_R is the temperature of the rigid body, $s_C(\cdot,r)$ is a prescribed distance function and k_e is the coefficient of heat exchange between elastoplastic body and rigid body.

2.1 Scales of Hilbert spaces

A collection of real Hilbert spaces $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ with scalar product $(\cdot,\cdot)_\alpha$ and norm $\|\cdot\|_\alpha$ is called *scale* of Hilbert spaces if the following is true:
(i) For any $\alpha > \beta$ the space H_α is continuously embedded into H_β, i.e. $H_\alpha \subset H_\beta$ and there exists a $c_1 > 0$ such that $\|h\|_\beta \le c_1\|h\|_\alpha$, $\forall\, h \in H_\alpha$, and H_α is dense in H_β ;
(ii) For any $\alpha > 0$ and $h \in H_\alpha$ the linear functional $(\cdot, h)_0$ on H_0 can be continuously extended to a linear continuous functional $(\cdot, h)_{-\alpha,\alpha}$ on $H_{-\alpha}$ satisfying $|(h', h)_{-\alpha,\alpha}| \le \|h'\|_{-\alpha}\|h\|_\alpha$, $\forall\, h' \in H_{-\alpha}$, $\forall\, h \in H_\alpha$. Any linear continuous functional ℓ on H_α has the form $\ell(h) = (h', h)_{-\alpha,\alpha}$ with some $h' \in H_{-\alpha}$, i.e., $H_{-\alpha}$ is isomorphic to the space of linear continuous functionals on H_α. From (i) it follows that for any $\alpha \in (\beta, \gamma)$ the space H_α is *rigged* by H_β and H_γ, i.e., $H_\gamma \subset H_\alpha \subset H_\beta$ with dense and continuous embeddings.

Example 1

Suppose $\Omega \subset \mathbb{R}^m$ is a domain and N is an arbitrary natural number. $\{H_\alpha^{(N)}\}_{\alpha\in\mathbb{R}}$ is the *scale of fractional Sobolev spaces* such that $H_\ell^{(N)} = W^{\ell,2}(\Omega)\,, \ell = 0, 1, \dots, N$, with norms $\|u\|^2_{H_\alpha^{(N)}}$ given by

$$\int_\Omega (|u|^2 + \sum_{|\beta|=1}^{\alpha} |D^\beta u|^2) dx =: \|u\|^2_{W^{\alpha,2}}\,,$$

$$\text{if} \quad \alpha \geq 0 \text{ integer},$$

$$\|u\|^2_{W^{k,2}} + \sum_{|\beta|=k} \int_\Omega \int_\Omega \frac{|D^\beta u(x) - D^\beta u(y)|^2}{|x-y|^{k+2\lambda}} dxdy\,,$$

$$\text{if} \quad \alpha = k + \lambda > 0\,, k \geq 0 \text{ integer}, \lambda \in (0,1),$$

$$\sup_{\|v\|_{H_{-\alpha}^{(N)}}=1} |\int_\Omega u(x)v(x)dx|\,, \text{if} \quad \alpha < 0.$$

2.2 A simplified contact problem

Suppose $\Omega \subset \mathbb{R}^m$ is a bounded domain, $\partial\Omega$ is smooth, $u = u(x,t)$ and $\Theta = \Theta(x,t)$ are the displacement and the temperature in the elastic body satisfying the system

$$u_{tt} + 2\varepsilon u_t - \Delta u + \alpha u = \xi(t), \quad \xi(t) \in \varphi(\Theta(t)), \tag{8}$$

$$\Theta_t - \beta\Delta\Theta + u - \gamma\zeta(t) = 0, \quad \zeta(t) = g(\Theta(t)), \tag{9}$$

with $\alpha, \beta, \varepsilon, \gamma$ constants, and the boundary and initial conditions

$$u = 0\,, \ \Theta = 0 \quad \text{on } \partial\Omega \times (0,T) \tag{10}$$

$$u(\cdot,0) = u_0(\cdot)\,, \ \dot{u}(\cdot,0) = u_1(\cdot)\,, \Theta(\cdot,0) = \Theta_0 \text{ in } \Omega. \tag{11}$$

$\varphi : \mathbb{R} \to 2^{\mathbb{R}}$ and g : $\mathbb{R} \to \mathbb{R}$ are nonlinear maps satisfying

$$vg(v) - \xi^2 \geq 0,\ \forall v \in \mathbb{R},\ \forall \xi \in \varphi(v) \tag{12}$$

and $g = \phi'$, i.e. g has a Fréchet differentiable potential.

$\mathcal{A}$ is the self-adjoint positive-definite operator generated by $(-\Delta)$ with zero boundary conditions and having the domain
$\mathcal{D}(\mathcal{A}) = W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$. Introduce the spaces
$\mathcal{V}_0 = L^2(\Omega), \mathcal{V}_1 = \mathcal{D}(\mathcal{A}^{1/2})$ and $\mathcal{V}_2 = \mathcal{D}(\mathcal{A})$ with

$$(u, v)_s = (\mathcal{A}^{s/2}u, \mathcal{A}^{s/2}v)\ ,\ \forall u, v \in \mathcal{V}_s, s = 0, 1, 2\,, \tag{13}$$

as scalar product and $Y_s = \mathcal{V}_{s+1} \times \mathcal{V}_s\,,\ Z_s = \mathcal{V}_{s+1}\,, s = 0, 1\,,$ with the scalar product in Y_s given by

$$((u, v), (\bar{u}, \bar{v}))_s = (u, \bar{u})_{s+1} + (v, \bar{v})_s\,, \forall (u, v), (\bar{u}, \bar{v}) \in Y_s\,. \tag{14}$$

The weak form of (8), (9) is a *parameter-dependent hybrid system consisting* of a *variational inequality* and a *variational equality* of the type

$$(\dot{y} - A(q)y - B(q)\xi, \eta - y)_{Y_{-1},Y_1} + \Psi(\eta, q) - \Psi(y, q) \geq 0\,, \tag{15}$$

$$w(t) = C(q)y,\ \xi(t) \in \varphi(t, w(t), v(t), q)\,, \forall \eta \in L^2(0, T; Y_1), \quad \text{a.e. on } (0, T), \tag{16}$$

$$(\dot{z} - A_1(q)z - B_1(q)\,\zeta, \vartheta)_{Z_{-1},Z_1} = 0\,, \tag{17}$$

$$v(t) = C_1(q)z\,, \qquad \zeta(t) \in g(t, w(t), v(t), q)\,,$$
$$\forall \vartheta \in L^2(0, T; Z_1),\ \text{a.a. on } (0, T)\,. \tag{18}$$

Here $q \in Q$ is a parameter, (Q, d) is a metric space.
For any $q \in Q$ we assume that

$$A(q) \in \mathcal{L}(Y_1, Y_{-1}), B(q) \in \mathcal{L}(\Xi, Y_{-1}), C(q) \in \mathcal{L}(Y_{-1}, W),$$
$$\Psi(\cdot, q) : Y_1 \to \mathbb{R}_+, \varphi(\cdot, \cdot, \cdot, q) : \mathbb{R}_+ \times W \times \Upsilon \to 2^{\Xi},$$
$$A_1(q) \in \mathcal{L}(Z_1, Z_{-1}), B_1(q) \in \mathcal{L}(\mathcal{Z}, Z_{-1}), g(\cdot, \cdot, \cdot, q) : \mathbb{R}_+ \times W \times \Upsilon \to \mathcal{Z},$$
$$Y_1, Y_{-1}, Z_1, Z_{-1}, \Xi, W, \mathcal{Z}, \Upsilon \quad \text{are real Hilbert spaces.}$$

A pair $\{y(\cdot), z(\cdot)\} \in L^2(0, T; Y_1) \times L^2(0, T; Z_1)$ is said to be a *solution* of (15)-(18) on $(0, T)$ if $\{\dot{y}(\cdot), \dot{z}(\cdot)\} \in L^2(0, T; Y_{-1}) \times L^2(0, T; Z_{-1})$ and there exists a pair $\{\xi(\cdot), \zeta(\cdot)\} \in L^2(0, T; \Xi) \times L^2(0, T; \mathcal{Z})$ such that $\{y(\cdot), z(\cdot), \xi(\cdot), \zeta(\cdot)\}$ satisfies (15)-(18) for a.e. $t \in (0, T)$ and $\int_0^T \Psi(y(t), q)dt < +\infty$. We assume that for any $T > 0$ such solutions exist.

Definition 1

Suppose that $\{S_\alpha\}, \{\tilde{S}_\alpha\}, \{R_\alpha\}$ and $\{\tilde{R}_\alpha\}$ are scales of real Hilbert spaces (*observation* and *output spaces*, respectively) and $D_\alpha \in \mathcal{L}(Y_1, S_\alpha)$, $E_\alpha \in \mathcal{L}(\Xi, S_\alpha), \tilde{D}_\alpha \in \mathcal{L}(Z_1, \tilde{S}_\alpha), \tilde{E}_\alpha \in \mathcal{L}(\mathcal{Z}, \tilde{R}_\alpha)$, $M_\alpha \in \mathcal{L}(Y_1, R_\alpha), N_\alpha \in \mathcal{L}(\Xi, R_\alpha), \tilde{M}_\alpha \in \mathcal{L}(Z_1, \tilde{R}_\alpha)$ and $\tilde{N}_\alpha \in \mathcal{L}(\mathcal{Z}, \tilde{R}_\alpha)$ are scales of linear operators (*observation* and *output operators*, respectively).

If $\{y(\cdot), z(\cdot), \xi(\cdot), \zeta(\cdot)\}$ is a response of (15)-(18) and $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in \mathbb{R}$, are arbitrary scale parameters the function

$$s(\cdot, \alpha, \tilde{\alpha}) = (D_\alpha y(\cdot) + E_\alpha \xi(\cdot), \tilde{D}_{\tilde{\alpha}} z(\cdot) + \tilde{E}_{\tilde{\alpha}} \zeta(\cdot)) \tag{19}$$

is called *observation (measurement* or *time series)* and the function

$$r(\cdot, \beta, \tilde{\beta}) = (M_\beta y(\cdot) + N_p \xi(\cdot), \tilde{M}_{\tilde{\beta}} z(\cdot) + \tilde{N}_{\tilde{\beta}} \zeta(\cdot)), \tag{20}$$

is called *(unobservable) output* of (15)-(18).

Definition 1 (continued)

For two responses $\{y_i(\cdot), z_i(\cdot), \xi_i(\cdot), \zeta_i(\cdot)\}\,,\ i = 1, 2\,,$ (21)

of (15)-(18) and arbitrary scale parameters $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in \mathbb{R}$ we define the deviations

$$\Delta y(\cdot) = y_1(\cdot) - y_2(\cdot), \quad \Delta z(\cdot) = z_1(\cdot) - z_2(\cdot),$$
$$\Delta \xi(\cdot) = \xi_1(\cdot) - \xi_2(\cdot), \quad \Delta \zeta(\cdot) = \zeta_1(\cdot) - \zeta_2(\cdot)\,, \tag{22}$$

$$\Delta s(\cdot, \alpha)^2 = \|D_\alpha \Delta y(\cdot) + E_\alpha \Delta \xi(\cdot)\|^2_{S_\alpha}\,,$$
$$\Delta \tilde{s}(\cdot, \tilde{\alpha})^2 = \|\tilde{D}_{\tilde{\alpha}} \Delta z(\cdot) + \tilde{E}_{\tilde{\alpha}} \Delta \zeta(\cdot)\|^2_{\tilde{S}_{\tilde{\alpha}}}\,, \tag{23}$$

$$\Delta r(\cdot, \beta)^2 = \|M_\beta \Delta y(\cdot) + N_\beta \Delta \xi(\cdot)\|^2_{R_\beta}\,,$$
$$\Delta \tilde{r}(\cdot, \tilde{\beta})^2 = \|\tilde{M}_{\tilde{\beta}} \Delta z(\cdot) + \tilde{N}_{\tilde{\beta}} \Delta \zeta(\cdot)\|^2_{\tilde{R}_{\tilde{\beta}}}\,, \tag{24}$$

Definition 2

Suppose that $a > 0, b > 0 (a < b)$ and $t_1 > 0$ are numbers. The observation (19) is *determining for the bifurcation "loss of (a, b, t_1)-stability"* of the output (20) at $q = q^*$ if there exist continuous near q^* real-valued functions $\alpha(\cdot), \tilde{\alpha}(\cdot), \beta(\cdot)$ and $\tilde{\beta}(\cdot)$ with the properties:

a) For $q = q_1$ the observation (19) with $\alpha = \alpha(q_1), \tilde{\alpha} = \tilde{\alpha}(q_1)$ is *determining for the (a, b, t_1)-stability* of the output (20) with $\beta = \beta(q_1), \tilde{\beta} = \tilde{\beta}(q_1)$, i.e., there exists an $\varepsilon_1 = \varepsilon_1(q_1) > 0$ such that for arbitrary two responses (21) and their deviations (22) - (24) which satisfy

$$\Delta\, r(0, \beta(q_1))^2 + \Delta\, \tilde{r}(0, \tilde{\beta}(q_1))^2 < a \tag{25}$$

the observation property

$$\int_0^{t^*} [\Delta\, s(t, \alpha(q_1))^2 + \Delta\, \tilde{s}(t, \tilde{\alpha}(q_1))^2] dt < \varepsilon_1 \tag{26}$$

for a time $t^* \in (0, t_1)$ implies the output property $\Delta r(t, \beta(q_1))^2 + \Delta \tilde{r}(t, \tilde{\beta}(q_1))^2 < b\,,\ \forall\, t \in (0, t_1)$.

Definition 2 (continued)

b) For $q = q_2$ the observation (19) with $\alpha = \alpha(q_2), \tilde{\alpha} = \tilde{\alpha}(q_2)$ is *determining for the (a, b, t_1)-instability* of the output (20) with $\beta = \beta(q_2), \tilde{\beta} = \tilde{\beta}(q_2)$, i.e., there exists an $\varepsilon_2 = \varepsilon_2(q_2) > 0$ such that for arbitrary two responses (21) and their deviations (22) – (24) which satisfy (25) the observation property

$$\int_0^{t^*} [\Delta\, s(t, \alpha(q_2))^2 + \Delta\, \tilde{s}(t, \tilde{\alpha}(q_2))^2] dt \geq \varepsilon_2$$

for a time $t^* \in (0, t_1)$ implies the output property

$$\Delta\, r(t^*, \beta(q_2))^2 + \Delta\, \tilde{r}(t^*, \tilde{\beta}(q_2))^2 \geq b\,.$$

Definition 3

Suppose that $q \in Q$ is arbitrary and $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in \mathbb{R}, a > 0$ are arbitrary numbers. The observation (19) is *determining* for the *a-convergence* of the output (20) if for any two responses (21) of (15) – (18) and their deviations (22) – (24) from

$$\int_t^{t+1} [\Delta\, s(\tau, \alpha)^2 + \Delta\, \tilde{s}(\tau, \tilde{\alpha})^2] d\tau \to 0 \tag{27}$$

$$\text{for} \quad t \to +\infty \quad \text{it follows that}$$

$$\limsup_{t \to +\infty} [\Delta\, r(t, \beta)^2 + \Delta \tilde{r}(t, \tilde{\beta})^2] \leq a\,. \tag{28}$$

4.1 Description of the uncertainty nonlinear part

Consider the system (15) – (18) with arbitrary but fixed $q \in Q$. Suppose that $F(\cdot,\cdot,q)$ and $G(\cdot,\cdot,q)$ are quadratic forms on $Y_1 \times \Xi$. The *class* $\mathfrak{N}(F,G)$ *of nonlinearities* for (15) consists of all set-valued maps

$$\varphi(\cdot,\cdot,\cdot,q) : \mathbb{R}_+ \times W \times \Upsilon \to 2^{\Xi} \tag{29}$$

satisfying the following property: For any sufficiently large $t_0, T, 0 < t_0 < T$, and any pairs of functions $y_1(\cdot), y_2(\cdot) \in L^2(0,T;Y_1), z_1(\cdot), z_2(\cdot) \in L^2(0,T;Z_1)$ and $\xi_1(\cdot), \xi_2(\cdot) \in L^2(0,T;\Xi)$ with

$$\xi_i(t) \in \varphi(t, C(q)y_i(t), C_1(q)z_i(t), q), \quad i = 1,2, \quad \text{a.a. } t \in [0,T], \tag{30}$$

and

$$\|C_1(q)z_i(t)\|_\Upsilon \le \Delta, \quad i = 1,2, \quad \text{a.a. } t \in [t_0,T], \tag{31}$$

where $\Delta > 0$ is a small number depending on the second subsystem (17), (18), it follows that

$$F(y_1(t) - y_2(t), \xi_1(t) - \xi_2(t), q) \ge 0 \quad \text{a.a. } t \in [t_0,T]\,. \tag{32}$$

There exist a continuous function $\Phi : W \to \mathbb{R}$ (*generalized potential*) and numbers $\lambda = \lambda(q) > 0$ and $\gamma = \gamma(q) > 0$ such that

$$\int_s^t G(y_1(\tau) - y_2(\tau), \xi_1(\tau) - \xi_2(\tau), q) d\tau$$
$$\geq \frac{1}{2}[\Phi(C(q)y_1(t) - C(q)y_2(t)) - \Phi(C(q)y_1(s) - C(q)y_2(s))]$$
$$+\lambda \int_s^t \Phi(C(q)y_1(\tau) - C(q)y_2(\tau)) d\tau \quad \text{for all} \quad s, t \in [t_0, T], s \leq t,$$

and

$$\Phi(C(q)y_1(t) - C(q)y_2(t)) \geq \gamma \| C(q)y_1(t) - C(q)y_2(t) \|_W^2,$$
$$\text{a.a.} \quad t \in [t_0, T]. \tag{33}$$

4.2 Assumptions for the existence of determining observers

Let $T > 0$ be an arbitrary number, $L^2(0, T; Y_j)\,, j = 1, 0, -1$, measurable spaces with norm $\|y(\cdot)\|_{2,j} = (\int_0^T \|y(t)\|_j^2 dt)^{1/2}$. Let $\mathfrak{W}_T$ be the space of functions $y(\cdot) \in L^2(0, T; Y_1)$ for which $\dot{y}(\cdot) \in L^2(0, T; Y_{-1})$ equipped with the norm

$$\|y(\cdot)\|_{\mathfrak{W}_T} = (\|y(\cdot)\|_{2,1}^2 + \|\dot{y}(\cdot)\|_{2,-1}^2)^{1/2} \tag{34}$$

(**A1**) There exists a number $\lambda = \lambda(q) > 0$ such that for any $T > 0$ and any element $f \in L^2(0, T; Y_{-1})$ the problem

$$\dot{y} = (A(q) + \lambda I)y + f(t), y(0) = y_0, \tag{35}$$

is *well-posed*, i.e., for arbitrary $y_0 \in Y_0, f(\cdot) \in L^2(0, T; Y_{-1})$ there exists a unique solution $y(\cdot) \in \mathfrak{W}_T$ satisfying (36) and depending continuously on the initial data, i.e., $\|y(\cdot)\|_{\mathfrak{W}_T}^2 \le c_1\|y_0\|_0^2 + c_2\|f(\cdot)\|_{2,-1}^2$, where $c_1 > 0$ and $c_2 > 0$ are some constants.

(**A1**) (continued)

Furthermore, any solution of $\dot{y} = (A(q) + \lambda I)y$, $y(0) = y_0$, is exponentially decreasing for $t \to +\infty$, i.e., there exist constants $c_3 > 0$ and $\varepsilon > 0$ such that $\|y(t)\|_0 \leq c_3 e^{-\varepsilon t}\|y_0\|_0, t > 0$.

(**A2**) There exists a number $\lambda = \lambda(q) > 0$ such that the operator $A(q) + \lambda I \in \mathcal{L}(Y_1, Y_{-1})$ is *regular*, i.e., for any $T > 0, y_0 \in Y_1, z_T \in Y_1$ and $f \in L^2(0, T; Y_0)$ the solutions of the *direct problem*

$$\dot{y} = (A(q) + \lambda I)y + f(t), y(0) = y_0,$$

and of the associated *dual problem*

$$\dot{z} = -(A(q) + \lambda I)^* z + f(t), z(T) = z_T,$$

are strongly continuous in t in the norm of Y_1.

(**A3**) There exist numbers $\lambda = \lambda(q) > 0, \delta = \delta(q) > 0$ and $\alpha = \alpha(q)$ such that the following two properties hold:

$$\text{a)} \quad F^c(y,\xi,q) + G^c(y,\xi,q) - \delta\|D^c_\alpha y + E^c_\alpha \xi\|^2_{S^c_\alpha} \leq 0\,,$$
$$\forall (y,\xi) \in Y^c_1 \times \Xi^c \exists\, \omega \in \mathbb{R} : i\omega y = (A^c(q) + \lambda I^c)y + B^c(q)\xi\,; \qquad (36)$$

b) The functional

$$J(y(\cdot),\xi(\cdot)) = \int_0^\infty [F^c(y(\tau),\xi(\tau),q) + G^c(y(\tau),\xi(\tau),q) - \delta\|D^c_\alpha y(\tau) + E^c_\alpha \xi(\tau)\|^2_{S^c_\alpha}]\, d\tau$$

is bounded from above on the set

$$\mathfrak{M}_{y_0} = \{y(\cdot),\xi(\cdot) : \dot{y} = (A^c(q) + \lambda I^c)y + B^c(q)\xi,$$
$$y(0) = y_0, y(\cdot) \in \mathfrak{W}^c_\infty\,,\ \xi(\cdot) \in L^2(0,\infty;\Xi^c\} \quad \text{for any} \quad y_0 \in Y^c_0.$$

Here $F^c, G^c, D^c_\alpha, E^c_\alpha, A^c, I^c, B^c, S^c_\alpha, \mathfrak{W}^c_\infty, \Xi^c$ denote the usual complexification of quadratic forms, linear operators and Hilbert spaces, respectively.

Theorem 1

Suppose that there exist numbers $\lambda = \lambda(q) > 0, \delta = \delta(q) > 0$ and $\alpha = \alpha(q)$ such that the assumptions (**A1**) – (**A3**)are satisfied. Suppose also that for any solutions of (15) – (18) there are a time $t_0 > 0$ and a number $\Delta > 0$ such that (31) is fulfilled for any $T > t_0$. Then the observation

$$s(\cdot) = (D_\alpha y(\cdot) + E_\alpha \xi(\cdot), 0) \tag{37}$$

is determining for the output a-convergence in (15), (18) with respect to the output

$$r(\cdot) = w(\cdot) = C(q)y(\cdot)\,, \tag{38}$$

where $a > 0$ is a certain number depending on $\Psi(\cdot, q)$ in (15).

4.3 Completeness defect of the observation operators

The frequency-domain condition (**A3**) depends on embedding properties of the Sobolev spaces under consideration. Assume, for example, that $G \equiv 0, E_\alpha = 0$ and $F(y, \xi, q) = q_1 \|y\|_0^2 - q_2 \|y\|_1^2, (y, \xi) \in Y_0 \times \Xi$, where q_1 and q_2 are certain real constants and $q = (q_1, q_2) \in Q$. In order to verify (36) we introduce the frequency-domain characteristic $\chi(i\omega, q) = (i\omega I^c - A_\lambda^c(q))^{-1} B^c(q)$ for $\omega \in \mathbb{R}$ s.t. $i\omega \in \rho(A_\lambda^c(q))$, where $A_\lambda^c(q) = A^c(q) + \lambda I^c$. The frequency-domain condition (36) is satisfied if

$$q_1 \|\chi(i\omega, q)\xi\|_{Y_0^c}^2 - q_2 \|\chi(i\omega, q)\xi\|_{Y_1^c}^2 - \delta \|D_\alpha^c \chi(i\omega, q)\xi\|_{S_\alpha^c}^2 \leq 0,$$
$$\forall \xi \in \Xi^c, \forall \omega \in \mathbb{R} : i\omega \in \rho(A_\lambda^c(q)). \tag{39}$$

Suppose that from the embedding $Y_1^c \subset Y_0^c \subset Y_{-1}^c$ and the properties of D_α we have the a priori estimate

$$\|v\|^2_{Y_0^c} \leq c_1 \|v\|^2_{Y_1^c} + c_2 \varepsilon_{D_\alpha^c} \|D_\alpha^c v\|^2_{S_\alpha^c} \ , \ \forall v \in Y_1^c, \tag{40}$$

where $c_1 > 0$ and $c_2 > 0$ are certain constants and

$$\varepsilon_{D_\alpha^c} = \varepsilon_{D_\alpha^c}(Y_1^c, Y_0^c) = \sup\{\|w\|_{Y_0^c} : w \in Y_1^c, D_\alpha^c w = 0\,, \|w\|_{Y_1^c} \leq 1\}$$

is the *completeness defect* of the observation operator D_α^c with respect to the embedding $Y_1^c \subset Y_0^c$. It follows from (40) that the frequency-domain condition (39) is satisfied if

$$q_1 c_1 \|\chi(i\omega, q)\xi\|^2_{Y_1^c} - q_2 \|\chi(i\omega, q)\xi\|^2_{Y_1^c} + q_1 c_2 \varepsilon_{D_\alpha^c} \|D_\alpha^c \chi(i\omega, q)\xi\|^2_{S_\alpha^c} - \delta \|D_\alpha^c \chi(i\omega, q)\xi\|^2_{S_\alpha^c} \leq 0, \quad \forall \xi \in \Xi^c, \quad \forall \omega \in \mathbb{R} : i\omega \in \rho(A_\lambda^c(q))\,. \tag{41}$$

For (41) it is sufficient that

$$q_1 c_1 - q_2 \leq 0 \quad \text{and} \quad q_1 c_2 \varepsilon_{D_\alpha^c} - \delta \leq 0\,. \tag{42}$$

The inequalities (42) describe a subset in the space of parameters of the variational inequality and of the observation operator. The second condition from (42) is always satisfied if $\varepsilon_{D_\alpha^c}$ is sufficiently small. Suppose that $D_\alpha y = (\ell_1(y), \ldots, \ell_k(y))$, where $\ell_i : Y_1 \to \mathbb{R}$, $i = 1, \ldots, k$, are continuous linear functionals and $Y_1 = W^{s,2}(\Omega)$, $Y_0 = W^{\sigma,2}(\Omega)$ with $s > \sigma$. Then $\varepsilon_{D_\alpha^c} \approx c_1(\frac{c_2}{k})^{s-\sigma}$, i.e., the completeness defect of the observation operator D_α depends on the smoothness properties of the embedding $Y_1^c \subset Y_0^c$.

Let us consider the hybrid system (15) – (18) with $\Psi \equiv 0$ as a first order variational equation with a set-valued nonlinearity. For this we define the new variables

$$\mathbf{y} = (y, z), \quad \mathbf{w} = (w, z), \quad \boldsymbol{\xi} = (\xi, \zeta), \quad \boldsymbol{\eta} = (\eta, \vartheta), \tag{43}$$

the product spaces

$$\mathcal{Y}_i = Y_i \times Z_i\,, \; i = 1, 0, -1, \quad \mathcal{W} = W \times \Upsilon, \quad \mathcal{U} = \Xi \times \mathcal{Z}\,, \tag{44}$$

the parameter-dependent operator matrices

$$\mathcal{A}(q) = \begin{bmatrix} A(q) & 0 \\ 0 & A_1(q) \end{bmatrix}, \quad \mathcal{B}(q) = \begin{bmatrix} B(q) \\ B_1(q) \end{bmatrix}, \quad \mathcal{C}(q) = [C(q), C_1(q)]\,, \tag{45}$$

and the nonlinear set-valued map

$$\boldsymbol{\varphi}(\cdot,\cdot,q) = (\varphi(\cdot,\cdot,\cdot,q), \quad g(\cdot,\cdot,\cdot,q)) : \mathbb{R}_+ \times \mathcal{W} \to 2^{\Xi} \times \mathcal{Z}\,. \tag{46}$$

Thus we can write the coupled system (15) – (18) as first order variational equation with set-valued nonlinearity in $\mathcal{Y}_{-1}$ as

$$\dot{\mathbf{y}} = \mathcal{A}(q)\mathbf{y} + \mathcal{B}(q)\boldsymbol{\xi}\,, \tag{47}$$

$$\mathbf{w}(t) = \mathcal{C}(q)\mathbf{y}(t), \quad \boldsymbol{\xi}(t) \in \varphi\,(t, \mathbf{w}(t), q). \tag{48}$$

The scales of observation resp. output spaces for (47), (48) are

$$\mathcal{S}_{\boldsymbol{\alpha}} = S_\alpha \times \tilde{S}_{\tilde{\alpha}}, \quad \mathcal{R}_{\boldsymbol{\alpha}} = R_\alpha \times \tilde{R}_{\tilde{\alpha}}, \quad \boldsymbol{\alpha} = (\alpha, \tilde{\alpha}) \in \mathbb{R}^2\,, \tag{49}$$

the scales of observation resp. output operators are

$$\mathcal{D}_{\boldsymbol{\alpha}} = \begin{bmatrix} D_\alpha & 0 \\ 0 & \tilde{D}_{\tilde{\alpha}} \end{bmatrix}, \quad \mathcal{E}_{\boldsymbol{\alpha}} = \begin{bmatrix} E_\alpha & 0 \\ 0 & \tilde{E}_{\tilde{\alpha}} \end{bmatrix}, \quad \mathcal{M}_{\boldsymbol{\alpha}} = \begin{bmatrix} M_\alpha & 0 \\ 0 & \tilde{M}_{\tilde{\alpha}} \end{bmatrix},$$

$$\mathcal{N}_{\boldsymbol{\alpha}} = \begin{bmatrix} N_\alpha & 0 \\ 0 & \tilde{N}_{\tilde{\alpha}} \end{bmatrix}. \tag{50}$$

It is clear that

$$\mathcal{D}_{\alpha} \in \mathcal{L}(\mathcal{Y}_1, \mathcal{S}_{\alpha}), \quad \mathcal{E}_{\alpha} \in \mathcal{L}(\mathcal{U}, \mathcal{S}_{\alpha}), \quad \mathcal{M}_{\alpha} \in \mathcal{L}(\mathcal{Y}_1, \mathcal{R}_{\alpha}), \\ \mathcal{N}_{\alpha} \in \mathcal{L}(\mathcal{U}, \mathcal{R}_{\alpha}), \quad \alpha \in \mathbb{R}^2 . \tag{51}$$

If $\{\mathbf{y}(\cdot), \xi(\cdot)\}$ is a response of (47), (48) and $\alpha, \beta \in \mathbb{R}^2$ are arbitrary scale parameters the function

$$\mathbf{s}(\cdot, \alpha) = \mathcal{D}_{\alpha}\mathbf{y}(\cdot) + \mathcal{E}_{\alpha}\,\xi\,(\cdot) \tag{52}$$

is the observation and

$$\mathbf{r}(\cdot, \beta) = \mathcal{M}_{\beta}\mathbf{y}(\cdot) + \mathcal{N}_{\beta}\xi(\cdot) \tag{53}$$

is the output of (47), (48).

Definition 4

Suppose that $\mathcal{F}$ und $\mathcal{G}$ are quadratic forms on $\mathcal{Y}_1 \times \mathcal{U}$. The class of nonlinearities $\mathfrak{N}(\mathcal{F}, \mathcal{G})$ for (47), (48) defined by $\mathcal{F}(\cdot, \cdot, q)$ and $\mathcal{G}(\cdot, \cdot, q)$ consists of all maps (46) such that the following conditions are satisfied: For any $T > 0$ and any two functions $\mathbf{y}(\cdot) \in L^2(0, T; Y_1)$ and $\boldsymbol{\xi}(\cdot) \in L^2(0, T; \mathcal{U})$ with

$$\boldsymbol{\xi}(t) \in \varphi(t, \mathcal{C}(q)\mathbf{y}(t), q), \quad \text{a.a. } t \in [0, T], \tag{54}$$

it follows that

$$\mathcal{F}(\mathbf{y}(t), \boldsymbol{\xi}(t), q) \geq 0, \quad \text{a.a. } t \in [0, T], \tag{55}$$

and there exists a continuous function $\Phi : \mathcal{Y}_1 \to \mathbb{R}$ such that

$$\int_s^t \mathcal{G}(\mathbf{y}(\tau), \boldsymbol{\xi}(t), q) d\tau \geq \Phi(\mathcal{C}(q)\mathbf{y}(t)) - \Phi(\mathcal{C}(q)\mathbf{y}(s)) \tag{56}$$

$$\text{for all} \quad 0 \leq s < t \leq T.$$

In the sequel we need the following assumptions for any $q \in Q$:

(A4) The operator $\mathcal{A}(q) \in \mathcal{L}(\mathcal{Y}_1, \mathcal{Y}_{-1})$ is regular, i.e., for any $T > 0$, $\mathbf{y}_0 \in \mathcal{Y}_1, \Psi_T \in \mathcal{Y}_1$ and $\mathbf{f} \in L^2(0, T; \mathcal{Y}_0)$ the solutions of the direct problem

$$\dot{\mathbf{y}} = \mathcal{A}(q)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(0) = \mathbf{y}_0, \quad \text{a.a.} \quad t \in [0, T],$$

and of the dual problem

$$\dot{\Psi} = -\mathcal{A}^*(q)\Psi + \mathbf{f}(t), \quad \Psi(T) = \Psi_T, \quad \text{a.a.} \quad t \in [0, T],$$

are strongly continuous in t in the norm of $\mathcal{Y}_1$.

(A5) The pair $(\mathcal{A}(q), \mathcal{B}(q))$ is L^2-controllable, i.e., for arbitrary $\mathbf{y}_0 \in \mathcal{Y}_0$ there exists a control $\xi(\cdot) \in L^2(0, \infty; \mathcal{U})$ such that the problem

$$\dot{\mathbf{y}} = \mathcal{A}(q)\mathbf{y} + \mathcal{B}(q)\xi, \quad \mathbf{y}(0) = \mathbf{y}_0$$

is well-posed on $[0, +\infty)$.

Definition 5

The variational equation (47), (48) is said to be *absolutely dichotomic in the class $\mathfrak{N}(\mathcal{F}, \mathcal{G})$ with respect to the output* $\mathbf{r}(\cdot, \beta)$ from (53) if for any response $\{\mathbf{y}(\cdot), \xi(\cdot)\}$ of (47), (48) with $\mathbf{y}(0) = \mathbf{y}_0, \xi(0) = \xi_0$ the following is true:
Either $\mathbf{y}(\cdot)$ is unbounded on $[0, \infty)$ in the $\mathcal{Y}_0$-norm or $\mathbf{y}(\cdot)$ is bounded in $\mathcal{Y}_0$ in this norm and there exist constants c_1 and c_2 (which depend only on $\mathcal{A}(q), \mathcal{B}(q)$ and $\mathfrak{N}(\mathcal{F}, \mathcal{G})$) such that

$$\|\mathcal{M}_\beta \mathbf{y}(\cdot) + \mathcal{E}_\beta \xi(\cdot)\|^2_{2,\mathcal{R}_\beta} \le c_1(\|\mathbf{y}_0\|^2_{\mathcal{Y}_0} + c_2).$$

Theorem 2

Suppose that $\varphi \in \mathfrak{N}(\mathcal{F}, \mathcal{G})$ and that for the operators $\mathcal{A}(q)$ and $\mathcal{B}(q)$ the assumptions **(A4)** and **(A5)** are satisfied. Suppose also that there exists a $\mu > 0$ such that the frequency-domain condition

$$\mathcal{F}^c(\mathbf{y}, \boldsymbol{\xi}, q) + \mathcal{G}^c(\mathbf{y}, \boldsymbol{\xi}, q) - \mu \|\mathcal{M}^c_{\boldsymbol{\beta}} \mathbf{y} + \mathcal{E}^c_{\boldsymbol{\beta}} \boldsymbol{\xi}\|^2_{\mathcal{R}_{\boldsymbol{\beta}}} \leq 0 ,$$

$$\forall (\mathbf{y}, \boldsymbol{\xi}) \in \mathcal{Y}^c_1 \times \mathcal{U}^c : \exists \omega \in \mathbb{R} \quad \text{with} \quad i\omega \mathbf{y} = \mathcal{A}^c(q)\mathbf{y} + \mathcal{B}^c(q)\xi$$

is satisfied and the functional

$$J(\mathbf{y}(\cdot), \boldsymbol{\xi}(\cdot), q) = \int_0^\infty [\mathcal{F}^c(\mathbf{y}(\tau), \boldsymbol{\xi}(\tau), q) + \tag{57}$$

$$\mathcal{G}^c(\mathbf{y}(\tau), \boldsymbol{\xi}(\tau), q) - \mu \|\mathcal{M}_{\boldsymbol{\beta}^c} \mathbf{y}(\tau) + \mathcal{E}^c_{\boldsymbol{\beta}} \boldsymbol{\xi}(\tau)\|^2_{\mathcal{R}^c_{\boldsymbol{\beta}}}] d\tau$$

Theorem 2 (continued)

is bounded from above on the set

$$\mathfrak{M}_{\mathbf{y}_0} = \{\mathbf{y}(\cdot), \boldsymbol{\xi}(\cdot) : \dot{\mathbf{y}} = \mathcal{A}^c(q)\mathbf{y} + \mathcal{B}^c(q)\boldsymbol{\xi}, \mathbf{y}(0) = \mathbf{y}_0 \,, \\ \mathbf{y}(\cdot) \in \mathfrak{W}^c_\infty \,, \quad \boldsymbol{\xi}(\cdot) \in L^2(0, \infty; \mathcal{U}^c)\}$$

for any $\mathbf{y}_0 \in \mathcal{Y}_0^c$. Assume additionally that any potential Φ from the class $\mathfrak{N}(\mathcal{F}, \mathcal{G})$ is nonnegative and there exists a constant $c > 0$ such that

$$\Phi(\mathcal{C}(q)\mathbf{y}) \le c\|\mathbf{y}\|^2_{\mathcal{Y}_0} \,, \quad \forall \mathbf{y} \in \mathcal{Y}_0 .$$

Then the equation (47), (48) is absolutely dichotomic in the class $\mathfrak{N}(\mathcal{F}, \mathcal{G})$ with respect to the output $\mathbf{r}(\cdot, \beta)$ from (53).

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Thank you
for your attention!