Bifurcations of invariant measures in discrete-time parameter dependent cocycles

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1. Basic tools for cocycle theory

Let (Q,d) be a complete metric space

A **base flow** $(\{\tau^t\}_{t\in\mathbb{R}},(Q,d))$ is defined by a continuous mapping $\tau:\mathbb{R}\times Q\to Q, (t,q)\mapsto \tau^t(q)$ satisfying

1) $\tau^\circ(\cdot)=\mathrm{id}_Q$,

2) $\tau^{t+s}(\cdot)=\tau^t(\cdot)\circ\tau^s(\cdot)$ for each $t,s\in\mathbb{R}$;

A **cocycle over the base flow** $(\{\tau^t\}_{t\in\mathbb{R}},(Q,d))$ is defined by the pair $(\{\varphi^t(q,\cdot)\}_{\substack{t\in\mathbb{R}\\ q\in Q}},(M,\rho))$, where (M,ρ) is a metric space and

1) $\varphi^t(q,\cdot):M\to M,\quad \forall\, t\in\mathbb{R},\quad \forall\, q\in Q$

2) $\varphi^o(q,\cdot)=\mathrm{id}_M,\quad \forall\, q\in Q,$

3) $\varphi^{t+s}(q,\cdot)=\varphi^t(\tau^s(q),\varphi^s(q,\cdot)),\quad \forall\, t,s\in\mathbb{R},\quad \forall\, q\in Q$.

Shortly we denote the **cocycle over the base flow** by (τ,φ). If $q\in Q\mapsto Z(q)\subset M$ is a map, we call $\widehat{Z}=\{Z(q)\}_{q\in Q}$ a **nonautonomous** set. The nonautonomous set $\widehat{Z}=\{Z(q)\}_{q\in Q}$ is said to be **invariant** for the cocycle (τ,φ) if

$$\varphi^t(q,Z(q))=Z(\tau^t(q)) \text{ for all } t\in\mathbb{R} \text{ and } q\in Q.$$

Rokhlin (1964); Kloeden, Schmalfuss (1997)

2. Hausdorff dimension estimates for invariant sets of cocycles

Suppose H is a separable Hilbert space, $K\subset H$ is a compact set, $L\in\mathcal{L}(H)$

$$\alpha_k(L)=\sup_{\substack{M\subset H\\ \dim M=k}}\ \inf_{\substack{u\in M\\ \|u\|=1}}\|Lu\|\,,\qquad k=1,2,...$$

$$\omega_k(L)=\begin{cases}\alpha_1(L)\cdot\ldots\cdot\alpha_k(L)\,, & \text{for}\quad k>0\\ 1\,, & \text{for}\quad k=0\,.\end{cases}$$

Suppose $d \geq 0$ is an arbitrary number. It can be represented as $d = d_0 + s$, where $d_0 \in \{0, 1, \cdots, n-1\}$ and $s \in [0, 1]$. Now we put

$$\omega_d(L) := \begin{cases} \omega_{d_0}(L)^{1-s}\omega_{d_0}(L)^{1+s}\,, & \text{for} \quad d > 0\,, \\ 1\,, & \text{for} \quad d = 0 \end{cases}$$

and we call $\omega_d(L)$ the **singular value function** of L of order d.

Boichenko, Leonov and Reitmann (2005)

Suppose (τ, φ) is a cocycle:

$$\begin{aligned} \tau^t : Q \to Q\,, \\ \varphi^t(\cdot,\cdot) : Q \times H \to H\,, \\ H \text{ is a Hilbert space}\,. \end{aligned}$$

Assumptions:

(A1) The nonautonomous set $\widehat{Z} = \{Z(q)\}_{q \in Q}$ is invariant for the cocycle (φ, τ).

(A2) For each $q \in Q$ and $t > 0$ let $\partial_2\, \varphi^t(q, \cdot) : H \to H$ be the Fréchet differential of $\varphi^t(q, \cdot)$ w.r.t. the second argument u, which has the following properties:

a) For each $\varepsilon > 0$ and $t > 0$ the function

$$g_\varepsilon(t, q) := \sup_{\substack{u,v \in Z(q) \\ 0 < \|v-u\| \leq \varepsilon}} \frac{\|\varphi^t(q,v) - \varphi^t(q,u) - \partial_2\, \varphi^t(q,u)(v-u)\|}{\|v-u\|}$$

is bounded on Q and converges to zero as $\varepsilon \to 0$.

b)

$$\sup_{q \in Q}\ \sup_{u \in Z(q)} \|\partial_2\, \varphi^t(q,u)\|_{op} < \infty$$

Theorem 1 (Reitmann, Slepukhin; 2011) *Suppose that the assumptions* **(A1)** *and* **(A2)** *are satisfied and the following conditions hold:*

1) There exists a compact set $\widetilde{K} \subset H$ *such that*

$$\overline{\bigcup_{q \in Q} Z(q)} \subset \widetilde{K} \,.$$

2) There exists a continuous function $\kappa : Q \times H \to \mathbb{R}_+$*, a time* $s > 0$ *and a number* $d > 0$ *such that*

$$Z(q) \subset Z(\tau^s(q))$$

and

$$\sup_{(q,u) \in Q \times \widetilde{K}} \frac{\kappa(\tau^s(q), \varphi^s(q,u))}{\kappa(q,u)} \, \omega_d(\partial_2 \, \varphi^s(q,u)) < 1 \qquad (1)$$

Then $\dim_H Z(q) \leq d \,, \quad \forall q \in Q \,.$

Stochastic version: Crauel, Flandoli (1998)

3. Invariant measures for cocycles

Let $(Q, \mathfrak{A}, \mu)$ be a probability space. A **metric dynamical system** (MDS) is given by a map $\tau^{(\cdot)}(\cdot) : \mathbb{Z} \times Q \to Q$. For fixed time this is a family of measurable maps which satisfies the group property

1) $\tau^0 = \mathrm{id}_Q \,;$ 2) $\tau^{t+s} = \tau^t \circ \tau^s, \forall\, t, s \in \mathbb{Z}.$

$\{\tau^t\}$ is assumed to be measure preserving, i.e., $\tau^t(\mu) = \mu$. Suppose that $(M, \mathfrak{B})$ is a measurable space. A **cocycle over the MDS** is given by a map $\varphi : \mathbb{Z}_+ \times Q \times M \to M$ which is for fixed time a $(\mathfrak{A} \otimes \mathfrak{B}, \mathfrak{B})$-measurable mapping and satisfies for $s, t \in \mathbb{Z}_+$ and almost all $q \in Q$ and $u \in M$ the relations

$$\varphi^\circ(q,u) = u \,; \quad \varphi^{t+s}(q,u) = \varphi^t(\tau^s(q), \varphi^s(q,u)).$$

It is possible to write the cocycle as a skew product flow $(q, u) \mapsto (\tau^t(q), \varphi^t(q, u)) =: \widehat{\varphi}^t(q, u)$.

An **invariant measure** $\widehat{\mu}$ for the cocycle (τ, φ) is a probability measure on $Q \times M$ which is invariant w.r.t. the skew product, i.e. $\forall t \in \mathbb{Z}_+$ $\widehat{\varphi}^t(\widehat{\mu}) = \widehat{\mu}$ and has the marginal $\pi_Q \widehat{\mu} = \mu$ where $\pi_Q : Q \times M \to Q$ is the projection. We can characterize invariant measures by their disintegration $\widehat{\mu}(d(q, u)) = \widehat{\mu}_q(du)\mu(dq) = \widehat{\mu}(q, du)d\mu(q)$. The **Perron-Frobenius operator** P is defined by

$$P\widehat{\mu}(q, Z(q)) := \widehat{\mu}(q, \varphi^{-1}(q, Z(\tau(q)))), \quad q \in Q,$$

where $\varphi^{-1}(q, Z(\tau(q)))$ is the preimage under $\varphi = \varphi^1$ of the set $Z(\tau(q))$.

Arnold (1998); Imkeller, Kloeden (2003)

Example 1 (Baladi, Viana; 1996) $\widehat{\varphi} : \widehat{I} \to \widehat{I}, \widehat{I} = \bigcup_{k \geq 0} (\{k\} \times B_k)$ with $B_0 = I$ the unit interval, $\{B_k\}$ subsets of $I, \widehat{\varphi}(k, u) = (k + 1, \varphi(u))$ a tower construction, where $\varphi : I \to I$ admits an invariant measure μ absolutely continuous w.r.t. m.

Introduce a cocycle $\kappa : \widehat{I} \to [0, \infty)$ and the Perron-Frobenius operator

$$P(\widehat{g})(k, y) = \sum_{\widehat{\varphi}(l,x)=(k,y)} \frac{\kappa(l, x)}{\kappa(k, y)} \frac{\widehat{g}(l, x)}{|\varphi'(x)|} \tag{2}$$

acting at the Banach space $BV(\widehat{I})$ of functions $\widehat{g} : \widehat{I} \to \mathbb{R}$ s. th.

$$\|\widehat{g}\|_{BV} = \mathrm{var}\, \widehat{g} + \sup |\widehat{g}| + \int |\widehat{g}|\kappa dx < \infty.$$

If ϱ is an eigenfunction of P associated to the eigenvalue 1 then $\widehat{\mu} = \varrho\kappa dx$ is an invariant measure for $\widehat{\varphi}$. Suppose $\widehat{\varphi}$ is invertible. Then (2) reduces with $q = k, u = x$ to

$$P(\widehat{g})(\widehat{\varphi}(q, u)) = \frac{\kappa(q, u)}{\kappa(\widehat{\varphi}(q, u))} \frac{\widehat{g}(q, u)}{|\varphi'(u)|}.$$

For the existence of an invariant measure we need

$$\frac{\kappa(q,u)}{\kappa(\widehat{\varphi}(q,u))}\,\frac{1}{|\varphi'(u)|} = 1 \tag{3}$$

or $$\frac{\kappa(\widehat{\varphi}(q,u))}{\kappa(q,u)}|\varphi'(u)| = 1\,, \quad \forall (q,u) \in Q \times I\,. \tag{4}$$

For $d = n$ and $s = 1$ we have $\omega_n(\partial_2\,\varphi^1(q,u)) = |\det \partial_2\,\varphi^1(q,u)|$. Thus if we consider (1) as equality this condition coincides with (4).

4. The Perron-Frobenius operator on rigged Hilbert spaces

Given a Hilbert space H. A subspace $\mathcal{H} \subset H$ is chosen such that the following holds:

1) $\mathcal{H}$ has a topology $\mathcal{T}$ with respect to which it is a locally convex vector space;

2) $(\mathcal{H}, \mathcal{T})$ is continuously and densely embedded into H ;

3) $(\mathcal{H}, \mathcal{T})$ is complete and barrelled.

The triplet $\mathcal{H} \subset H \subset \mathcal{H}'$ where $\mathcal{H}'$ denotes the topological dual of $\mathcal{H}$ is called **rigged Hilbert space** or **Gelfand triplet**. Suppose $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$. Then the **adjoint** w.r.t. H is the operator $A^+ \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ which is given by

$$(A\eta, \vartheta) = (A^+\vartheta, \eta) \quad \forall\, \eta, \vartheta \in \mathcal{H}$$

where $(\cdot,\cdot)$ is the pairing w.r.t. H.The operator A is **selfadjoint** w.r.t. H if $A^+ = A$;

A is **normal** if $A^+A = AA^+$

Berezansky (1968); Gelfand, Vilenkin (1964)

Example 2 Consider the microwave heating problem

$$\left.\begin{aligned}
& w_{tt} - w_{xx} + \sigma(\theta) w_t = 0\,, \\
& \theta_t - \theta_{xx} = \sigma(\theta) w_t^2\,,\ 0 < x < 1\,,\ t > 0\,, \\
& w(0,t) = f_1(t),\ w(1,t) = f_2(t)\,, \\
& \theta(0,t) = \theta(1,t) = 0\,,\ t > 0\,, \\
& w(x,0) = w_0(x)\,,\ w_t(x,0) = w_1(x)\,, \\
& \theta(x,0) = \theta_0(x)\,,\ 0 < x < 1\,.
\end{aligned}\right\} \qquad (5)$$

Assumptions:

(A3) σ is locally Lipschitz on $(0,+\infty)$. There exist constants $0 < \sigma_0 \leq \sigma_1$ such that $\sigma_0 < \sigma(z) \leq \sigma_1$ for any $z > 0$. σ is monotonically decreasing on $(0,+\infty)$.

(A4) $w_0 \in H^1(0,1)\,,\ w_1 \in L^2(0,1)\,,\ \theta_0 \in W_3^2(0,1)\,,$
$\theta_0 \geq 0$ a.e. on $(0,1)$.

(A5) $f_1, f_2 \in C^2(\mathbb{R})$ and there exists a constant c such that the functions $|f_1'|\,, |f_2'|\,, |f_1''|\,, |f_2''|$ are bounded on $\mathbb{R}$ by the constant c.

Manoranjan, Yin (2006): For any $T > 0$ there exists a global weak solution $(w(x,t)\,,\ \theta(x,t))$ of the problem (5) such that
$w \in L^\infty(0,T;H^1(0,1))\,,\ \theta \in W_3^{2,1}((0,1)\times(0,T))$

Introduce for $t \geq 0$ and $x \in (0,1)$ the new functions

$$f(x,t) = f_1(t)(1-x) + f_2(t)x\,,\ \psi(x,t) = w(x,t) - f(x,t).$$

We get

$$\left.\begin{aligned}
& \psi_{tt} - \psi_{xx} + \sigma(\theta)\psi_t = f_{tt}(x,t) - f_t(x,t)\sigma(\theta)\,, \\
& \theta_t - \theta_{xx} = \sigma(\theta)(\psi_t + f_t)^2\,,\ 0 < x < 1\,,\ t > 0\,, \\
& \psi(0,t) = \psi(1,t) = 0\,,\ \theta(0,t) = \theta(1,t) = 0\,, t > 0\,, \\
& \psi(x,0) = \psi_0(x) = w_0(x) - f(x,0)\,, \\
& \psi_t(x,0) = w_1(x) - f_t(x,0)\,, \\
& \theta(x,0) = \theta_0(x)\,,\ 0 < x < 1\,.
\end{aligned}\right\} \qquad (6)$$

Define $M = H_0^1(0,1) \times L^2(0,1) \times (W_3^2(0,1) \cap \{\theta | \theta \geq 0\,, \text{a.e.}\})$

$\|(\psi, v, \theta)\|_M^2 = \|\psi_x\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2 + \|\theta\|_{L^2(0,1)}^2\,.$

Introduce $Q = \mathbb{R}\,,\ \tau^t(s) = t + s\,,\ \varphi^t(s, u_0) = u(t + s, s, u_0)$ where $u(t, s, u_0) = (\psi(\cdot, t), \psi_t(\cdot, t)\,, \theta(\cdot, t))$ is a solution of (6) with $u(s, s, u_0) = u_0$.

Theorem 2 (Kalinin, Reitmann, Yumaguzin, 2011)

System (6) generates a cocycle (τ, φ) which admits a global pull-back -B attractor.

Define $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} w_t(x,t) \\ w(x,t) \\ \theta(x,t) \end{pmatrix}$ and consider the evolution system

$$\frac{dy}{dt} = Ay + B\xi\,,\ y(0) = y_0 \tag{7}$$

on rigged Hilbert spaces $Y_1 \subset Y_0 \subset Y_{-1}$ with Ξ a Hilbert space, $A : Y_1 \to Y_{-1}\,,\ B : \Xi \to Y_{-1}$ linear operators and ξ a nonlinear function.

Let φ be an endomorphism of a measure space $(M, \mathfrak{B})$. The evolution operator U is given by the **Koopman operator**

$$(Ug)(x) = g(\varphi(x))\,,$$

where g is a square-integrable function.

The adjoint is the **Perron-Frobenius operator** P.
(Lasota, Mackey; 1985)

5. Parameter-dependent cocycles and bifurcations

Let $(Q_\alpha, \mathfrak{A}_\alpha, \mu_\alpha)$ be a family of probability spaces depending on a parameter $\alpha \in \mathcal{A}$, where $(\mathcal{A}, \rho_\mathcal{A})$ is a metric space. A **parametric metric dynamical system** (PMDS) is given by a family of maps $\tau_\alpha^t(\cdot) : Q_\alpha \to Q\alpha$ which are measurable and satisfy the properties

1) $\tau_\alpha^0 = \mathrm{id}_{Q_\alpha}$; 2) $\tau_\alpha^{t+s} = \tau_\alpha^t \circ \tau_\alpha^s\,,\ t,s \in \mathbb{Z}\,.$

$\{\tau_\alpha^t\}_{\substack{t\in\mathbb{Z}\\ \alpha\in\mathcal{A}}}$ is assumed to be measure preserving, i.e. $\tau_\alpha^t(\mu_\alpha) = \mu_\alpha\,,\ t \in \mathbb{Z}\,,\ \alpha \in \mathcal{A}$. Suppose that $(M, \mathfrak{B})$ is an other measurable space. A **parametric cocycle** over the PMDS is given by a family of parameter dependent maps $\varphi_\alpha^t(\cdot) : Q_\alpha \times M \to M$ which are $(\mathfrak{A}_\alpha \otimes \mathfrak{B}, \mathfrak{B})$ measurable maps and satisfy the cocycle property. We write the parametric cocycle as a parametric skew product system

$$(q,u) \in Q_\alpha \times M \mapsto (\tau_\alpha^t(q), \varphi_\alpha^t(q,u)) =: \widehat{\varphi}_\alpha^t(q,u)\,,\ t \in \mathbb{Z},\ \alpha \in \mathcal{A}.$$

A **family of invariant measures** $\{\widehat{\mu}_\alpha\}_{\alpha\in\mathcal{A}}$ for the parametric cocycle is a family of probability measures on $Q \times M$ which is invariant w.r.t the parametric skew product, i.e., $\widehat{\varphi}_\alpha^t(\widehat{\mu}_\alpha) = \widehat{\mu}_\alpha$ and $\pi_{Q_\alpha}\widehat{\mu}_\alpha = \mu_\alpha\,,\ \alpha \in \mathcal{A}$.

A parameter value α_0 is called a **bifurcation point** of the family of invariant measures $\{\widehat{\mu}_\alpha\}_{\alpha\in\mathcal{A}}$ if this family is not structurally stable at α_0, i.e., if in any neighborhood of α_0 there are parameter values $\alpha \in \mathcal{A}$ s. th. $\{\widehat{\varphi}_{\alpha_0}^t\}$ and $\{\widehat{\varphi}_\alpha^t\}$ are not topologically equivalent.

Arnold (1998)

Example 3 The Rényi map $\varphi_\alpha : [0,1] \to [0,1]$ is given by $\varphi_\alpha(x) = \alpha x \bmod 1$ with $\alpha > 1$. This map generates a metric dynamical system $(\{\varphi_\alpha^t\}, m)$, where m denotes the Lebesgue measure on the unit interval.

The Koopman operator U_α for $\alpha \in \mathbb{N}$ is given by

$$(U_\alpha g)(x) = \alpha^{-1} \sum_{i=0}^{\alpha-1} g(\varphi_{\alpha,i}(x))\,,$$

where $\varphi_{\alpha,i}$ is the inverse of the Rényi map on its i-th interval of monotonicity.

(Bandtlow, Antoniou and Suchanecki, 1997)

The associated Perron-Frobenius operator P_α of this map

$$P_\alpha : L^2(m) \to L^2(m)$$

is given by

$$P_\alpha \eta = \frac{d}{dm} \int\limits_{\varphi_\alpha^{-1}(\cdot)} \eta \, dm \,,$$

where $\frac{d}{dm}$ is the Radon-Nikodym derivative w.r.t. m. As positive function spaces $\mathcal{H}$ we can use spaces which are densely and continuously embedded in $L^2(m)$:

- Banach spaces $\mathcal{E}_c (c > 0)$ of entire functions of exponential type c ;
- Fréchet spaces $\mathcal{H}(D_r)\,(r > 1)$ of functions analytic in the open disk with radius r ;
- Fréchet space C^∞ of infinitely differentiable functions on the closed unit interval.

For $c < c'$ and $r < r'$ we have

$$\mathcal{E}_c \hookrightarrow \mathcal{E}_{c'} \hookrightarrow \mathcal{H}(D_{r'}) \hookrightarrow \mathcal{H}(D_r) \hookrightarrow C^\infty \hookrightarrow L^2(m) \,.$$

The map under perturbations: $\varphi_\alpha(q,u) = \varphi_\alpha(u) + q$. Consider the skew product system $\widehat{\varphi}_\alpha : Q \times I \to Q \times I =: \widehat{I}$ with $\widehat{\varphi}^k(q,u) = (\tau^k(q), \varphi^k(q,u))$ and the associated function spaces $L^r(\widehat{I})$.

References

[1] L. Arnold, Random Dynamical Systems. Springer, 1998.

[2] V. Baladi and M. Viana, *Strong stochastic stability and rate of mixing for unimodal maps.* Ann. Sci. École Norm. Sup., 29, 483 - 517, 1996.

[3] F. Bandtlow, I. Antoniou and Z. Suchanecki, *Resonances of dynamical systems and Fredholm-Riesz operators on rigged Hilbert spaces.* Computers Math. Appl. 34 (2-4), 95 - 102, 1997.

[4] Yu. M. Berezansky, Expansions in Eigenfunctions of Selfadjoint Operators. Amer. Math. Soc. Transl. 17, Providence, R. I., 1968.

[5] V. A. Boichenko, G.A. Leonov and V. Reitmann, Dimension Theory for Ordinary Differential Equations. Vieweg-Teubner Verlag, Wiesbaden, 2005.

[6] H. Crauel and F. Flandoli, *Hausdorff dimension of invariant sets for random dynamical systems.* Journal of Dynamics and Differential Equations, 10, 449 - 474, 1998.

[7] I. Gelfand and N. Vilenkin, Generalized functions, Vol. 4: Applications of Harmonic Analysis. Rigged Hilbert Spaces, Academic Press, New York, 1964.

[8] L. Glass, M.R. Guevera and A. Shrier, *Universal bifurcations and the classification of cardiac arrhythmias.* Ann. N. Y. Acad. Sci., 504, 168 - 178, 1987.

[9] P. Imkeller and P. Kloeden, *On the computation of invariant measures in random dynamical systems.* Stochastics and Dynamics, 3 (2), 247 - 265, 2003.

[10] Yu. Kalinin, V. Reitmann and N. Yumaguzin, *Asymptotic behavior of Maxwell's equation in one-space dimension with thermal effect.* Discrete and Continuous Dynamical Systems, 2, 754 - 762, 2011.

[11] P. E. Kloeden and B. Schmalfuss, *Nonautonomous systems, cocycle attractors and variable time-step discretization.* Numerical Algorithms, 14 (1-3), 141 - 152, 1997.

[12] A. Lasota and M. C. Mackey, Probabilistic Propertie of Deterministic Systems. Cambridge University Press, Cambridge, 1985.

[13] R. V. Manoranjan and H. M. Yin, *On two-phase Stefan problem arising from a microwave heating process.* Discrete and Continuous Dynamical Systems, Serie A, 15, 1155 - 1186, 2006.

[14] V. Reitmann, Dynamical Systems, Attractors and Estimates of their Dimension. St. Petersburg State University Press, St. Petersburg, 2012 (Russian).

[15] V. Reitmann and A. Slepukhin, *Upper Hausdorff dimension estimates for negatively invariant sets of local cocycles.* Vestn. St. Petersburg State Univers., Ser. 1, vyp. 4, 61 - 70, 2011.

[16] V. Rokhlin, *Exact endomorphisms of a Lebesgue space.* Am. Math. Soc. Transl., 39 (2), 1 - 36, 1964.