Determining functionals for cocycles and application to the microwave heating problem

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1. Introduction

Let (Q,d) be a metric space called the *base space*. The pair $(\{\tau^t\}_{t\in\mathbb{R}},(Q,d))$ where $\tau^t : Q \to Q$ for each $t \in \mathbb{R}$ is called the *base flow* if

$$\begin{aligned} &\tau^0 = id_Q, \\ &\tau^t \circ \tau^s = \tau^{t+s} \quad \forall t,s \in \mathbb{R}. \end{aligned} \tag{1}$$

Let (M,ρ) be an other metric space (*phase space*).

Definition 1 *The pair* $(\{\varphi^t(q,\cdot)\}_{t\in\mathbb{R}_+,q\in Q},(M,\rho))$ *where* $\varphi^t(q,\cdot) : M \to M$ *for each* $t \in \mathbb{R}_+, q \in Q$ *is called a cocycle over the base flow* $(\{\tau^t\}_{t\in\mathbb{R}},(Q,d))$ *if*

$$\begin{aligned} &\varphi^0(q,\cdot) = id_M \ \forall q \in Q, \\ &\varphi^{t+s}(q,\cdot) = \varphi^t(\tau^s(q),\varphi^s(q,\cdot)) \ \forall q \in Q,\ \forall t,s \in \mathbb{R}_+. \end{aligned} \tag{2}$$

For brevity the cocycle $(\{\varphi^t(q,\cdot)\}_{t\in\mathbb{R}_+,q\in Q},(M,\rho))$ over the base flow $(\{\tau^t\}_{t\in\mathbb{R}},(Q,d))$ will be denoted by (φ,τ).

Define the space $W = Q \times M$ with the metric

$$\tilde{\rho}((q_1,u_1),(q_2,u_2)) = \max\{d(q_1,q_2),\rho(u_1,u_2)\}, \quad (q_i,u_i) \in Q \times M, i = 1,2$$

and the family of mappings $S^t : W \to W$, $t \in \mathbb{R}_+$, $S^t(q,u) = (\tau^t(q),\varphi^t(q,u))$.
The dynamical system $(\{S^t\}_{t\in\mathbb{R}_+},(W,\tilde{\rho}))$ is called *skew product*.

A *non-autonomous set* $\hat{\mathcal{C}} = \{\mathcal{C}(q)\}_{q\in Q}$ is a mapping $Q \to 2^M$. A nonautonomous set is called *bounded* *(closed, compact)* if for any $q \in Q$ the set $\mathcal{C}(q)$ is bounded (closed, compact) in M.

A bounded non-autonomous set $\hat{\mathcal{C}}$ is said to be a *globally B-pullback absorbing* set for (φ, τ) if for any $q \in Q$ and any bounded set $\mathcal{B} \subset M$ there exists a $T = T(q, \mathcal{B})$ such that $\varphi^t(\tau^{-t}(q, \mathcal{B})) \subset \mathcal{C}(q)$ for $t \geq T$.

A non-autonomous set $\hat{\mathcal{C}}$ is called *globally B-pullback attracting* for (φ, τ) if for any $q \in Q$ and any bounded set $\mathcal{B} \subset M$

$$\lim_{t \to +\infty} \text{dist}(\varphi^t(\tau^{-t}(q), \mathcal{B}), \mathcal{C}(q)) = 0,$$

where dist is the Hausdorff semidistance in (M, ρ).

A non-autonomous set $\hat{\mathcal{C}}$ is called *invariant (positively invariant)* for (φ, τ) if for any $q \in Q$ and $t \geq 0$ the equality $\varphi^t(q, \mathcal{C}(q)) = \mathcal{C}(\tau^t(q))$ (inclusion $\varphi^t(q, \mathcal{C}(q)) \subset \mathcal{C}(\tau^t(q))$) holds.

Definition 2 *A non-autonomous set is called a* global B-pullback attractor *for the cocycle* (φ, τ) *if it is compact, invariant and is globally B-pullback attracting.*

For the existence proof of a B-pullback attractor we will use the following theorem [Kloeden-Schmalfuss, 1987]:

Theorem 1 *Let the cocycle* (φ, τ) *have a compact globally B-pullback absorbing set* $\hat{\mathcal{C}} = \{\mathcal{C}(q)\}_{q \in Q}$*. Then* (φ, τ) *has a unique B-pullback attractor* $\hat{\mathcal{A}} = \{\mathcal{A}(q)\}_{q \in Q}$*, where for each* $q \in Q$

$$\mathcal{A}(q) = \cap_{t \in \mathbb{R}_+} \overline{\cup_{s \geq t, s \in \mathbb{R}_+} \varphi^s(\tau^{-s}(q), \mathcal{C}(\tau^{-s}(q))}.$$

2. Existence of a B-pullback attractor for the 1-dimensional microwave heating problem

The derivation of the 1-dimensional microwave heating problem is given in [H.-M. Yin et al., 2006].

Consider the initial-boundary problem

$$\begin{array}{lll} w_{tt} - w_{xx} + \sigma(\theta) w_t = 0, & 0 < x < 1, & t > 0 \\ \theta_t - \theta_{xx} = \sigma(\theta) w_t^2, & 0 < x < 1, & t > 0 \end{array} \tag{3}$$

$$\begin{array}{lll} w(0,t) = f_1(t), & w(1,t) = f_2(t), & t > 0 \\ \theta(0,t) = \theta(1,t) = 0, & & t > 0 \end{array} \tag{4}$$

$$\begin{array}{lll} w(x,0) = w_0(x), & w_t(x,0) = w_1(x), & 0 < x < 1 \\ \theta(x,0) = \theta_0(x), & & 0 < x < 1 \end{array} \tag{5}$$

where $\theta(x,t)$ is the temperature, $w(x,t)$ is the time integral of the nonzero component of the electric field, $\sigma(\theta)$ is the electric conductivity, $f_1(t), f_2(t)$ are the external perturbations of the electric field.

Assumptions:

(A1.1) σ is locally Lipschitz on $(0, +\infty)$;

(A1.2) There exist constants $0 < \sigma_0 \leq \sigma_1$ such that $\sigma_0 \leq \sigma(z) \leq \sigma_1$ for any $z > 0$;

(A1.3) σ is monotone decreasing.

(A2) $w_0 \in L^2(0,1), w_1 \in L^2(0,1), \theta_0 \in L^2(0,1), \theta_0 \geq 0$ a.e. on $(0,1)$.

(A3) f_1, f_2 are $C^2(\mathbb{R})$ and there exists a constant c such that the functions $|f_1'|, |f_2'|, |f_1''|, |f_2''|$ are bounded on $\mathbb{R}$ by c.

Modification of the existence theorem for weak solutions from [H.-M. Yin et al., 2006] for the 1-dimensional case:

Theorem 2 *For any $T > 0$ there exists a global weak solution $(w(x,t), \theta(x,t))$ of the problem (3)-(5) such that $w \in C([0,T]; L^2(0,1))$, $\theta \in L^2(0,T; H^1(0,1)) \cap C([0,T]; L^2(0,1))$.*

Additional assumption:

(A4) The weak solution is unique.

Denote $f(x,t) = f_1(t)(1-x) + f_2(t)x$ and $\psi(x,t) = w(x,t) - f(x,t)$ and introduce the system with homogeneous boundary conditions, i.e.

$$\begin{aligned} &\psi_t = \zeta - f_t, \\ &\zeta_t = \psi_{xx} - \sigma(\theta)\zeta, \\ &\theta_t = \theta_{xx} + \sigma(\theta)(\psi_t + f_t)^2, \quad 0 < x < 1, \quad t > 0 \end{aligned} \tag{6}$$

with initial and boundary conditions

$$\psi(0,t) = \psi(1,t) = 0, \quad \theta(0,t) = \theta(1,t) = 0, \quad t > 0 \tag{7}$$

$$\begin{aligned} &\psi(x,0) = \psi_0(x) = w_0(x) - f(x,0), && 0 < x < 1 \\ &\zeta(x,0) = \zeta_0(x) = w_1(x) - f_t(x,0), && 0 < x < 1 \\ &\theta(x,0) = \theta_0(x), && 0 < x < 1. \end{aligned} \tag{8}$$

Transformed assumption (A2):

(A2') $\psi_0 \in H_0^1(0,1), \zeta_0 \in L^2(0,1), \theta_0 \in L^2(0,1), \theta_0 \geq 0$ a.e. on $(0,1)$.

Introduction of the cocycle corresponding to the problem (6)-(8)

Define the metric space

$M = H_0^1(0,1) \times L^2(0,1) \times (L^2(0,1) \cap \{\theta : \theta \geq 0\})$ with the norm

$$\|(\psi,\zeta,\theta)\|_M^2 = \|\psi_x\|_{L^2(0,1)}^2 + \|\zeta\|_{L^2(0,1)}^2 + \|\theta\|_{L^1(0,1)}^2.$$

In our situation: $Q = \mathbb{R}$, $\tau^t(s) = t + s$,

$$\varphi^t(s, u_0) = u(t+s, s, u_0),$$

where $u(t,s,u_0)=(\psi(\cdot,t),\zeta(\cdot,t),\theta(\cdot,t))$ is the solution of (6)-(8) such that $u(s,s,u_0)=u_0=(\psi_0,\zeta_0,\theta_0)$.

From existence and uniqueness of the solution we conclude (I. Ermakov, Y. Kalinin, V. Reitmann, 2011):

Theorem 3 *The system (6)-(8) generates a cocycle* $(\{\varphi^t(s,\cdot)\}_{t\in\mathbb{R}_+,s\in\mathbb{R}},(M,\|\cdot\|_M))$ *over the base flow* $(\{\tau^t\}_{t\in\mathbb{R}},\mathbb{R})$.

Proof of the existence of an absorbing set:

- Lyapunov function for the 1st subsystem (damped wave equation);
- monotonicity methods for the 2nd subsystem (heat equation).

Damped wave equation. Consider the initial-boundary problem for the wave equation separately:

$$\psi_{tt}-\psi_{xx}+\sigma(x,t)\psi_t=f_{tt}-\sigma(x,t)f_t,\quad 0<x<1,\quad t>s \tag{9}$$

$$\psi(0,t)=\psi(1,t)=0,\quad t>s \tag{10}$$

$$\psi(x,s)=\psi_0,\quad \psi_t(x,s)=\psi_1,\quad 0<x<1 \tag{11}$$

where $s\in\mathbb{R}$. Here $\sigma(x,t)$ is a certain function.

Modified assumptions (A1)-(A3):

(A1.2*) There exist constants $0<\sigma_0\le\sigma_1$ such that $\sigma_0\le\sigma(x,t)\le\sigma_1$ for all $x\in(0,1),t\ge 0$.

(A2*) $\psi_0 \in H_0^1(0,1), \psi_1 \in L^2(0,1)$.

(A3*) The function $f(x,t)$ is C^1 in x, C^2 in t and there exists a constant $c > 0$ such that $|f_t| < c$, $|f_{xt}| < c$, $|f_{tt}| < c$ for any $x \in (0,1), t \in \mathbb{R}$.

Under the assumptions **(A1.2*)** - **(A3*)** the problem (9 - 11) has a unique solution $(\psi(\cdot,t), \psi_t(\cdot,t)) \in M_1 = H_0^1(0,1) \times L^2(0,1)$ [R. Temam, 1993].
For $(\psi,\zeta) \in M_1$ define

$$\|(\psi,\zeta)\|_{M_1}^2 = \|\psi_x\|_{L^2(0,1)}^2 + \|\zeta\|_{L^2(0,1)}^2$$

Write equation (9) as first order system

$$\begin{aligned} \psi_t &= \zeta - f_t, \\ \zeta_t &= \psi_{xx} - \sigma(x,t)\zeta \end{aligned} \tag{12}$$

with boundary and initial conditions

$$\psi(0,t) = \psi(1,t) = 0, \quad t > s \tag{13}$$

$$\psi(x,s) = \psi_0(x), \zeta(x,s) = \zeta_0(x), \quad 0 < x < 1 \tag{14}$$

Proposition 1 *For any $t > 0$ there exist $T > 0$, $c > 0$ such that $\|(\psi(\cdot,t;s), \zeta(\cdot,t;s))\|_{M_1} < c$ for any $s \le t - T$.*

Idea of the proof: Lyapunov functional on M_1

$$V(\psi,\zeta) = \|\psi_x\|^2 + 2\lambda(\psi,\zeta) + \|\zeta\|^2$$

where $\lambda > 0$ is a parameter. $\|\cdot\|$ and $(\cdot,\cdot)$ are in $L^2(0,1)$.

Denote $V(t) = V(\psi(\cdot,t), \zeta(\cdot,t))$.

We prove that there exist $\delta > 0$, $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$\frac{d}{dt}V(t) \le -\delta V(t) + c_1,$$

$$V(t) \le e^{-\delta(t-s)}V(s) + c_2,$$

for any t, s, $t \ge s$.

The nonlinear heat equation (2nd equation of (6))

General setting (A.A.Pankov, 1983):

Suppose that $E \subset H \subset E'$ is a Gelfand triple, i.e. $(E, \|\cdot\|_E)$ is a reflexive Banach space, H is a Hilbert space, E' is the space dual to E, E is continuously and densely embedded into H.

Suppose that $A(t) : E \to E'$ is a family of operators and $F : \mathbb{R} \to E'$ is a measurable function .
The operator $A(t) : E \to E'$ is monotone, i.e.

$$(A(t)u - A(t)v, u - v) \ge 0, \ \forall u, v \in E.$$

Here $(\cdot, \cdot)$ is the duality pairing on $E \times E'$, coinsiding on $E \times E$ with the scalar product in H.

Consider the evolution equation

$$\frac{du}{dt} + A(t)u = \tilde{f}(t). \tag{15}$$

Suppose that there is an $\alpha > 0$ such that

$$(A(t)u - A(t)v, u - v) \ge \alpha \|u - v\|^2 \ \forall u, v \in E. \tag{16}$$

Define the following function spaces:

- $C_b(\mathbb{R}, E)$ is the set of continuous functions $f : \mathbb{R} \to E$, for which $\sup_{t\in\mathbb{R}} \|f(t)\|_E$ is finite.

- $BS^p(\mathbb{R}, E), 1 \le p < \infty$ is the subspace in $L^p_{loc}(\mathbb{R}, E)$, consisting of functions with finite norm

$$\|f\|^p_{S^p} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p_E \, ds \right).$$

Consider the heat equation in the form

$$\theta_t - \theta_{xx} = \sigma(\theta) g(x,t).$$

Suppose that $g(x,t) \ge 0$ is measurable and uniformly bounded in t. We have $g(x,t) = (\psi_t(x,t) + f_t(x,t))^2$. For σ the assumptions (A1.1)-(A1.3) hold.

$\sigma(\theta) = \sigma_0 + \tilde{\sigma}(\theta)$, where σ_0 is from (A1.2). We get

$$\theta_t - \theta_{xx} - \tilde{\sigma}(\theta) g(x,t) = \sigma_0 g(x,t), \quad 0 < x < 1, \quad t > s \tag{17}$$

$$\theta(0,t) = \theta(1,t) = 0, \quad t > s \tag{18}$$

$$\theta(x,s) = \theta_0(x), \quad 0 < x < 1. \tag{19}$$

The initial-boundary problem (17)-(19) generates an evolution equation (15), where
$A(t)u = -u'' - g(x,t)\tilde{\sigma}(u)$ for $u \in E$ and $\tilde{f}(t) = \sigma_0 g(\cdot, t)$.

In our situation we have $E = H^1_0(0,1)$ and $H = L^2(0,1)$. Check condition (16). Let $u, v \in H^1_0(0,1)$, $\eta = u - v$. Then

$$(A(t)u - A(t)v, u-v) = (-\eta'', \eta) + (g(\cdot,t)(\tilde{\sigma}(v) - \tilde{\sigma}(u)), \eta) =$$

$$= (\eta', \eta') + (g(\cdot,t)(\tilde{\sigma}(v) - \tilde{\sigma}(u)), \eta) \ge \|\eta\|^2.$$

A.A. Pankov, 1983:

1. The Cauchy problem for equation (15) has a unique solution $u \in BS^2(\mathbb{R}, H_0^1(0,1)) \cap C_b(\mathbb{R}, L^2(0,1))$. For the equation (17) this means that there exists a constant c_1 such that $\|\theta(\cdot,t;s)\| \le c_1$ for any $t, s \in \mathbb{R}$, $s \le t$.

2. We have the estimate

$$\|\theta_1(\cdot,t;s) - \theta_2(\cdot,t;s)\| \le e^{-c_2(t-s)} \|\theta_{01} - \theta_{02}\|, \quad (20)$$

for $t > s$, where $\theta_i(x,t;s)$ is the solution of (17) with initial data θ_{0i} and initial time s.

The constant c_1 does not depend on initial data:

$$\theta(x,t;s) = \int_0^1 G(x,y;t,s)\theta_0(y)dy + \int_s^t \int_0^1 G(x,y;t,r)g(y,r)drdy,$$

where $G(x,y;t,r)$ is the corresponding Green's function which satisfies

$$|G(x,y;t,s)| \le \frac{c_3}{\sqrt{t-s}}.$$

The influence of initial data tends to zero for $t \to \infty$.

Make the initial time s tend to $-\infty$, which corresponds to the time shift in $g(x,t)$.

Proposition 2 *Let $\theta(\cdot,t;s)$ be the solution of (17)-(19). There exists a constant c such that for all t and $s \le t$ the inequality $\|\theta(\cdot,t;s)\| \le c$ holds where c does not depend on θ_0.*

From uniform boundedness in s of solutions of the wave equation and the heat equation we obtain

Theorem 4 *The cocycle* (φ, τ) *generated by problem* (6) – (8) *has a globally B-pullback absorbing set.*

Applying the Kloeden-Schmalfuss Theorem, we get

Theorem 5 *The cocycle* (φ, τ) *generated by problem* (6) – (8) *has a global B-pullback attractor.*

3. Determining functionals for cocycles

Physical meaning: Asymptotically finite-dimensional dynamics
C. Foias, G. Prodi, 1967
O. Ladyzhenskaya, 1975
I.D. Chueshov, 1998
I.D. Chueshov, J. Duan, B. Schmalfuss, 2001.

If the system has a global attractor, such functionals can give an approximation of the attractor.

Let $(\{S^t\}_{t\in\mathbb{R}_+}, (E, \|\cdot\|))$ be a dynamical system on Banach space $(E, \|\cdot\|)$.

Definition 3 *The set* $\{l_j\}_{j=1}^N$ *of linear continuous functionals on* E *is called a set of asymptotically determining functionals for the dynamical system* $(\{S^t\}_{t\in\mathbb{R}_+}, (E, \|\cdot\|))$ *if for any* $u_1, u_2 \in E$ *the condition*

$$\lim_{t\to+\infty} \left| l_j(S^t(u_1)) - l_j(S^t(u_2)) \right| = 0, \ j = 1, ..., N$$

implies

$$\lim_{t\to+\infty} \left\| S^t(u_1) - S^t(u_2) \right\| = 0.$$

Introduce the determining modes which are important examples of determining functionals.

Definition 4 *The determining modes for the dynamical system $(\{S^t\}_{t\in\mathbb{R}_+}, (H, (\cdot,\cdot)))$ on a Hilbert phase space $(H, (\cdot,\cdot))$ are determining functionals $l_j(\cdot) = (\cdot, e_j)$ where $\{e_j\}_1^N$ are some elements of H.*

The notion of pullback-asymptotically determining functionals for processes was introduced in [J.A. Langa, 2003]. We give a generalization for cocycles.

Definition 5 *The set $\{l_j\}_{j=1}^N$ of linear continuous functionals on Banach space $(M, \|\cdot\|)$ is called a set of pullback-asymptotically determining functionals for the cocycle $(\{\varphi^t(q,\cdot)\}_{q\in Q, t\in\mathbb{R}_+}, (M, \|\cdot\|))$ over the base flow $(\{\tau^t\}_{t\in\mathbb{R}}, (Q, d))$ if the condition*

$$\lim_{t\to+\infty} \left| l_j(\varphi^t(\tau^{-t}(q), u_1)) - l_j(\varphi^t(\tau^{-t}(q), u_2)) \right| = 0$$

for any $q \in Q$, $u_1, u_2 \in M$, $j = 1, .., N$ implies

$$\lim_{t\to+\infty} \left\| \varphi^t(\tau^{-t}(q), u_1) - \varphi^t(\tau^{-t}(q), u_2) \right\| = 0.$$

Let (φ, τ) be a cocycle on a Hilbert phase space H, π_1 be the projector from H onto a finite-dimensional subspace of H and π_2 be its complement.

Assumptions:

(H1) The non-autonomous set $\{\mathcal{C}(q)\}_{q\in Q}$ is positively invariant for (φ,τ).

(H2) For any $q \in Q$ there exists $\delta = \delta(q) \in (0,1)$ such that for all $s \geq 1, u, v \in \mathcal{C}(\tau^{-s}(q))$

$$\left\|\pi_2(\varphi^1(\tau^{-s}(q),u) - \varphi^1(\tau^{-s}(q),v))\right\| \leq \delta(q)\left\|u-v\right\|.$$

Let $a_1, a_2 : Q \to H$ be mappings such that $a_i(q) \in \mathcal{C}(q)$ for any $q \in Q$.

(H3) For any $\varepsilon > 0$, $t \geq 0$ there exists an $L = L(\varepsilon) \in \mathbb{N}$ such that for any $q \in Q$

$$\delta(q)^{2L}\left\|\varphi^{t-L}(q,a_1(q)) - \varphi^{t-L}(q,a_2(q))\right\|^2 < \varepsilon,$$

and $L(\varepsilon) \to \infty$ if $\varepsilon \to 0$.

The next theorem [I. Ermakov, Y. Kalinin, V. Reitmann, 2011] is a generalization of Theorem 14 from [J.A. Langa, 2003]

Theorem 6 *Let the assumptions (H1)-(H3) hold and suppose that there exists a $\beta > 0$ such that for any $q \in Q$*

$$\lim_{t\to+\infty}\left\|\pi_1(\varphi^t(\tau^{-t}(q),a_1(q)) - \varphi^t(\tau^{-t}(q),a_2(q)))\right\| \leq \beta.$$

Then

$$\lim_{t\to+\infty}\left\|\varphi^t(\tau^{-t}(q),a_1(q)) - \varphi^t(\tau^{-t}(q),a_2(q))\right\| \leq \beta. \quad (21)$$

Corollary 1 *Let there exist a* $\beta > 0$ *such that for all* $q \in Q, u, v \in H$

$$\lim_{t\to+\infty} \left\|\pi_1(\varphi^t(\tau^{-t}(q), u) - \varphi^t(\tau^{-t}(q), v))\right\| \leq \beta.$$

Then

$$\lim_{t\to+\infty} \left\|\varphi^t(\tau^{-t}(q), u) - \varphi^t(\tau^{-t}(q), v)\right\| \leq \beta.$$

This corollary gives the existence of pullback-asymptotically determining modes for a cocycle.

Now consider cocycles of a special type. Such cocycles are generated by the microwave heating problem.

Let (φ, τ) be a cocycle with phase space $E = E_1 \times E_2$ where E_1 is a Hilbert and E_2 is a Banach space, respectively.

φ has the form (φ_1, φ_2), i.e.

$$\varphi_1 : \mathbb{R}_+ \times Q \times E_1 \times E_2 \to E_1,$$

$$\varphi_2 : \mathbb{R}_+ \times Q \times E_1 \times E_2 \to E_2.$$

Let π_1 be the projector from E_1 onto a finite-dimensional subspace of E_1, π_2 be its orthogonal complement.

Let $a_1, a_2 : Q \to E$ be mappings such that $a_i(q) \in \mathcal{C}(q)$ for any $q \in Q$.

Modify assumptions (H2) and (H3) so that they hold for φ_1 instead of φ:

(H2*) For any $q \in Q$ there exists a $\delta = \delta(q) \in (0,1)$ such that for all $s \geq 1, u, v \in \mathcal{C}(\tau^{-s}(q))$ we have

$$\left\|\pi_2(\varphi_1^1(\tau^{-s}(q),u) - \varphi_1^1(\tau^{-s}(q),v))\right\|_{E_1} \leq \delta(q)\,\|u-v\|_E\,.$$

(H3*) For any $\varepsilon > 0$, $t \geq 0$ there exists an $L = L(\varepsilon) \in \mathbb{N}$ such that for any $q \in Q$

$$\delta(q)^{2L}\left\|\varphi_1^{t-L}(q,a_1(q)) - \varphi_1^{t-L}(q,a_2(q))\right\|_{E_1}^2 < \varepsilon,$$

and $L(\varepsilon) \to \infty$ if $\varepsilon \to 0$.

Theorem 7 *Suppose that*

1) The assumptions (H1),(H2),(H3*) hold for the cocycle* (φ,τ)*.*

2) The estimate

$$\left\|\varphi_2^t(\tau^{-t}(q),u_1,u_2) - \varphi_2^t(\tau^{-t}(q),v_1,v_2)\right\|_{E_2} \leq e^{-ct}\,\|u_2-v_2\|_{E_2}$$

holds with some constant $c > 0$ *for any* $t > 0$, $u_1, v_1 \in E_1$, $u_2, v_2 \in E_2$*.*

3) There exists a $\beta > 0$ *such that for any* $q \in Q$

$$\lim_{t\to+\infty}\left\|\pi_1(\varphi_1^t(\tau^{-t}(q),a_1(q)) - \varphi_1^t(\tau^{-t}(q),a_2(q)))\right\|_{E_1} \leq \beta.$$

Then

$$\lim_{t\to+\infty}\left\|\varphi^t(\tau^{-t}(q),a_1(q)) - \varphi^t(\tau^{-t}(q),a_2(q))\right\|_E \leq \beta. \tag{22}$$

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