Frequency-domain conditions for convergence to the stationary set in coupled PDEs

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1. Introduction

Suppose: Y_0 a real Hilbert space, $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ the scalar product resp. the norm on Y_0,
$A:\mathcal{D}(A)\to Y_0$ the generator of a C_0-semigroup on Y_0,
$Y_1:=\mathcal{D}(A)$.
For fixed $\beta\in\rho(A)\cap\mathbb{R}$ for any $y,\eta\in Y_1$ define

$$(y,\eta)_1 := ((\beta I - A)y,\ (\beta I - A)\eta)_0\ . \tag{1}$$

Y_{-1} is the completion of Y_0 with respect to the norm,
$\|y\|_{-1} := \|(\beta I - A)^{-1}y\|_0$ is the scalar product

$$(y,\eta)_{-1} := \left((\beta I - A)^{-1}y,\ (\beta I - A)^{-1}\eta\right)_0, \qquad \forall\, y,\eta\in Y_{-1}. \tag{2}$$

$Y_1\subset Y_0\subset Y_{-1}$ is a continuous embedding, i.e., for $\alpha=1,0$,
$Y_\alpha\subset Y_{\alpha-1}\,,\ \|y\|_{\alpha-1}\le c\|y\|_\alpha\,,\ \forall\, y\in Y_\alpha$.

(Y_1,Y_0,Y_{-1}) is called a *Gelfand triple*.
For any $y\in Y_0$ and $z\in Y_1$ we have

$$|(y,z)_0| = |(\beta I - A)^{-1}y,((\beta I - A)z)_0| \le \|y\|_{-1}\|z\|_1\,. \tag{3}$$

Extend $(\cdot,z)_0$ by continuity onto Y_{-1}

$$|(y,z)_0|\le\|y\|_{-1}\|z\|_1\ ,\quad \forall\, y\in Y_{-1},\forall\, z\in Y_1.$$

Denote this extension also by $(\cdot,\cdot)_{-1,1}$.
Consider the Bochner measurable functions in

$L^2(0,T;Y_j)\quad (j=1,0,-1)$

$$\|y(\cdot)\|_{2,j} := \left(\int_0^T \|y(t)\|_j^2\,dt\right)^{1/2}. \tag{4}$$

$\mathcal{L}_T$ is the space of functions $y\in L^2(0,T;Y_1)$, s.th. $\dot y\in L^2(0,T;Y_{-1})$.
$\mathcal{L}_T$ is equipped with the norm

$$\|y\|_{\mathcal{L}_T} := \left(\|y(\cdot)\|_{2,1}^2 + \|\dot y(\cdot)\|_{2,-1}^2\right)^{1/2}. \tag{5}$$

2. Evolutionary variational systems

Take $T > 0$ arbitrary and consider for a.a. $t \in [0, T]$ the evolutionary variational equation

$$(\dot{y} - Ay - B\xi - f(t), \eta - y)_{-1,1} = 0\,, \qquad \forall\, \eta \in Y_1 \quad (6)$$
$$y(0) = y_0 \in Y_0\,,$$
$$w(t) = Cy(t)\,, \quad \xi(t) = \varphi(t, w(t))\,, \quad (7)$$
$$\xi(0) = \xi_0\,,$$
$$z(t) = Dy(t) + E\,\xi(t)\,. \quad (8)$$

$C \in \mathcal{L}(Y_{-1}, W), D \in \mathcal{L}(Y_1, Z)$ and $E \in \mathcal{L}(\Xi, Z)$,
Ξ, W and Z are real Hilbert spaces, $Y_1 \subset Y_0 \subset Y_{-1}$ is a real Gelfand triple and $A \in \mathcal{L}(Y_0, Y_{-1}),\ B \in \mathcal{L}(\Xi, Y_{-1})$,
$\varphi : \mathbb{R}_+ \times W \to \Xi$, $f : \mathbb{R}_+ \to Y_{-1}$.

Denote by $\|\cdot\|_\Xi,\ \|\cdot\|_W,\ \|\cdot\|_Z$ the norm in Ξ, W resp. Z .

Definition 1 Any pair of functions $\{y\,(\cdot), \xi(\cdot)\}$ with $y \in \mathcal{L}_T$ and $\xi \in L^2_{\text{loc}}(0, \infty; \Xi)$ such that $B\xi \in \mathcal{L}_T$, satisfying (6), (7) almost everywhere on $(0, T)$, is called **solution of the Cauchy problem** $y\,(0) = y_0,\ \xi(0) = \xi_0$ defined for (6), (7) .

Assumptions:

(C1) The Cauchy-problem (6), (7) has for arbitrary $y_0 \in Y_0$ and $\xi_0 \subset \Xi$ at least one solution $\{y\,(\cdot), \xi(\cdot)\}$.

(C2) The nonlinearity $\varphi : \mathbb{R}_+ \times W \to \Xi$ is a function having the property that $\mathcal{A}(t) := -A - B\varphi(t, C\cdot) : Y_1 \to Y_{-1}$ is a family of monotone hemicontinuous operators such that the inequality

$$\|\mathcal{A}(t)y\|_{-1} \leq c_1\|y\|_1 + c_2\,, \quad \forall\, y \in Y_1\,,$$

is satisfied, where $c_1 > 0$ and $c_2 \in \mathbb{R}$ are constants not depending on $t \in [0, T]$.

For any $y \in Y_1$ and for any bounded set $U \subset Y_1$ the family of functions $\{(\mathcal{A}(t)\eta, y)_{-1,1}\,,\ \eta \in U\}$ is equicontinuous with respect to t on any compact subinterval of $\mathbb{R}_+$.

(C3) $f \in L^2_{\text{loc}}(\mathbb{R}_+; Y_{-1})$.

(C4) Consider only solutions y of (6),(7) for which $\dot{y}$ belongs to $L^2_{\text{loc}}(\mathbb{R}; Y_{-1})$.

Definition 2 Suppose F and G are quadratic forms on $Y_1 \times \Xi$. The **class of nonlinearities** $\mathcal{N}(F,G)$ defined by F and G consists of all maps $\varphi : \mathbb{R}_+ \times W \to \Xi$ such that for any $y(\cdot) \in L^2_{\text{loc}}(0,\infty; Y_1)$ with $\dot{y}(\cdot) \in L^2_{\text{loc}}(0,\infty; Y_{-1})$ and any $\xi(\cdot) \in L^2_{\text{loc}}(0,\infty; \Xi)$ with $\xi(t) = \varphi(t, Cy(t))$ for a.e. $t \geq 0$, it follows that $F(y(t), \xi(t)) \geq 0$ for a.e. $t \geq 0$ and for any such pair $\{y, \xi\}$) there exists a continuous functional $\Phi : W \to \mathbb{R}$ such that for any times $0 \leq s < t$ we have

$$\int_s^t G(y(\tau), \xi(\tau))d\tau \geq \Phi(Cy(t)) - \Phi(Cy(s)) \,.$$

3. Further assumptions

(F1) $A \in \mathcal{L}(Y_1, Y_{-1})$ is *regular*, i.e., for any $T > 0, y_0 \in Y_1$, $\psi_T \in Y_1$ and $f \in L^2(0,T; Y_0)$ the solutions of the direct problem

$$\dot{y} = Ay + f(t)\,,\ y(0) = y_0\,, \quad \text{a.a. } t \in [0,T]$$

and of the dual problem

$$\dot{\psi} = -A^*\psi + f(t)\,,\ \psi(T) = \psi_T\,, \quad \text{a.a. } t \in [0,T]$$

are strongly continuous in t in the norm of Y_1.
$A^* \in \mathcal{L}(Y_{-1}, Y_0)$ denotes the adjoint to A, i.e.,
$(Ay, \eta)_{-1,1} = (y, A^*\eta)_{-1,1}\,,\ \forall\, y, \eta \in Y_1\,.$

(F2) The pair (A, B) is L^2*-controllable*, i.e., for arbitrary $y_0 \in Y_0$ exists a control $\xi(\cdot) \in L^2(0, \infty; \Xi)$ such that the problem

$$\dot{y} = Ay + B\xi\,, \quad y(0) = y_0$$

is well-posed on the semiaxis $[0, +\infty)$, i.e., there exists a solution $y(\cdot) \in \mathcal{L}_\infty$ with $y\,(0) = y_0$.

(F3) $F(y, \xi)$ is an Hermitian form on $Y_1 \times \Xi$, i.e.,

$$F(y, \xi) = (F_1 y, y)_{-1,1} + 2\,\mathrm{Re}\,(F_2 y, \xi)_\Xi + (F_3\,\xi, \xi)_\Xi,$$

where

$$F_1 = F_1^* \in \mathcal{L}(Y_1, Y_{-1})\,,\ F_2 \in \mathcal{L}(Y_0, \Xi)\,,\ F_3 = F_3^* \in \mathcal{L}(\Xi, \Xi)\,.$$

Define the *frequency-domain condition* [Likhtarnikov and Yakubovich, 1976]

$$\alpha := \sup_{\omega, y, \xi}(\|y\|_1^2 + \|\xi\|_\Xi^2)^{-1} F(y, \xi)\,,$$

where the supremum is taken over all triples $(\omega, y, \xi) \in \mathbb{R}_+ \times Y_1 \times \Xi$ such that $i\omega y = Ay + B\xi$.

4. Absolute observation - stability of evolutionary equations

For a function $z(\cdot) \in L^2(\mathbb{R}_+; Z)$ we denote their norm by

$$\|z(\cdot)\|_{2,Z} := \left(\int_0^\infty \|z(t)\|_Z^2 \, dt \right)^{1/2} .$$

Definition 3 a) The equation (6), (7) is said to be **absolutely dichotomic** (i.e., in the class $\mathcal{N}(F,G)$) **with respect to the observation** z from (8) if for any solution $\{y(\cdot), \xi(\cdot)\}$ of (6), (7) with $y(0) = y_0$, $\xi(0) = \xi_0$ the following is true: Either $y(\cdot)$ is unbounded on $[0,\infty)$ in the Y_0-norm or $y(\cdot)$ is bounded in Y_0 in this norm and there exist constants c_1 and c_2 (which depend only on A, B and $\mathcal{N}(F,G)$ such that

$$\|Dy(\cdot) + E\xi(\cdot)\|_{2,Z}^2 \le c_1(\|y_0\|_0^2 + c_2) . \tag{9}$$

b) The equation (6), (7) is said to be **absolutely stable with respect to the observation** z from (8) if (9) holds for any solution $\{y(\cdot), \xi(\cdot)\}$ of (6), (7).

Definition 4 The equation (6)$-$(8) with $f \equiv 0$ is said to be **minimally stable**, i.e., there exists a bounded linear operator $K : Y_1 \to \Xi$ such that the operator $A + BK$ is stable, i.e. for some $\varepsilon > 0$

$$\sigma(A + BK) \subset \{s \in \mathbb{C} : \mathrm{Re}\, s \le -\varepsilon < 0\}$$

$$\text{with} \qquad F(y, Ky) \ge 0 , \qquad \forall\, y \in Y_1 , \tag{10}$$

$$\text{and} \qquad \int_s^t G(y(\tau), \; Ky(\tau))d\tau \ge 0 ,$$

$$\forall\, s,t : 0 \le s < t , \quad \forall\, y \in L^2_{\mathrm{loc}}(\mathbb{R}_+; Y_1) . \tag{11}$$

Theorem 1 Consider the evolution problem (6) $-$ (8) with $\varphi \in \mathcal{N}(F,G)$. Suppose that for the operators A^c, B^c the assumptions **(F1)** and **(F2)** are satisfied. Suppose also that there exist an $\alpha > 0$ such that with the transfer operator

$$\chi^{(z)}(s) = D^c(sI^c - A^c)^{-1}B^c + E^c \qquad (s \notin \sigma(A^c)) \tag{12}$$

the frequency-domain condition

$$F^c\left((i\omega I^c - A^c)^{-1}B^c\xi, \xi\right)$$
$$+ G^c\left((i\omega I^c - A^c)^{-1}B^c\xi, \xi\right) \leq -\alpha\|\chi^{(z)}(i\omega)\xi\|^2_{Z^c}$$
$$\forall\, \omega \in \mathbb{R} : i\omega \notin \sigma(A^c)\,, \quad \forall\, \xi \in \Xi^c$$

is satisfied and the functional

$$J(y(\cdot),\xi(\cdot)) := \int_0^\infty [F^c(y(\tau),\xi(\tau)) + G^c(y(\tau),\xi(\tau))$$
$$+ \alpha\|D^c y(\tau) + E^c\xi(\tau)\|^2_{Z^c}]\, d\tau$$

is bounded from above on any set

$$\mathbf{M}_{y_0} := \{y(\cdot),\xi(\cdot) : \dot{y} = Ay + B\xi \quad \text{on} \quad \mathbb{R}_+,$$
$$y(0) = y_0\,,\, y(\cdot) \in \mathcal{L}_\infty\,,\, \xi(\cdot) \in L^2(0,\infty;\Xi)\}\,.$$

Suppose further that the equation (6)$-$(8) with $f \equiv 0$ is minimally stable, i.e., (10) and (11) are satisfied with some operator $K \in \mathcal{L}(Y_1, \Xi)$ and that the pair $(A + BK, D + EK)$ is observable in the sense of Kalman, i.e., for any solution $y(\cdot)$ of

$$\dot{y} = (A + BK)y\,, \quad y(0) = y_0\,,$$

with $z(t) = (D + EK)y\,(t) = 0$ for a.a. $t \geq 0$ it follows that $y(0) = y_0 = 0$.
Then equation (6), (7) is absolutely stable with respect to the observation z from (8).

A. L. Likhtarnikov and V.A. Yakubovich, 1976
Reitmann, V. and H. Kantz, 2003

5. Example

Consider the coupled system of Maxwell's equation and heat transfer equation

$$\begin{cases} \Psi_{tt} + \sigma(x,\theta)\Psi_t - \Psi_{xx} = 0 \\ \theta_t - \theta_{xx} = \sigma(x,\theta)\Psi_t^2 \end{cases} \tag{13}$$

Initial-boundary conditions:

$$\begin{aligned} &\Psi(0,t) = \theta(0,t) = 0, \\ &\Psi(1,t) = \theta(1,t) = 0 \quad \forall t \in [0,T] \\ &\Psi(x,0) = \Psi_0(x), \Psi_t(x,0) = \Psi_1(x), \theta(x,0) = \theta_0(x), \quad \forall x \in \Omega \end{aligned} \tag{14}$$

Here $x \in \Omega, t \in [0,T], T > 0, \Omega = (0,1)$.

Energy inequality:

$$\begin{aligned} &\sup_{0\le t\le T} \int_0^1 [\Psi_t^2 + \Psi_x^2] dx + \int_0^T \int_0^1 \sigma(x,t,\theta)\Psi_t^2 dxdt \\ &\le C_1 + C_2 \int_0^T \int_0^1 |\theta| dxdt\,, \end{aligned}$$

where the constants C_1 and C_2 depend only on known data.

System in terms of operator equations in some function spaces:

$$y(x,t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \Psi_t(x,t) \\ \Psi(x,t) \\ \theta(x,t) \end{pmatrix}, \tag{15}$$

$$\xi(x,t) = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \sigma(x,\theta)\Psi_t(x,t) \\ \sigma(x,\theta)\Psi_t^2(x,t) \end{pmatrix}.$$

Let us define operators A, B from equation (6). Let Λ be the self-adjoint positiv operator, generated on $L^2(0,1)$ by the differential expression $\Lambda(v) = -v_{xx}$ and zero boundary conditions (14).

Consider the following spaces $Y_0 = L^2(0,1)\times L^2(0,1)\times L^2(0,1)$, $Y_1 = W^{1,2}(0,1)\times W^{1,2}(0,1)\times W^{1,2}(0,1)$ and $\Xi = L^2(0,1)\times L^2(0,1)$ as defined in Section 1. Thus (**F1**) of Theorem 1 is satisfied.

Then operators A and B are defined as follows:

$$A = \begin{bmatrix} -\sigma_0 I & \Lambda & 0 \\ -I & 0 & 0 \\ 0 & 0 & -\Lambda \end{bmatrix}, B = \begin{bmatrix} -I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}, \qquad (16)$$

Here constant $\sigma_0 > 0$ is derived from decomposition $\sigma(x,\theta) = \sigma_0 + \bar{\sigma}(x,\theta)$.

Finally, system (13) can be written in terms of the operator equation

$$\frac{dy}{dt} = Ay + B\xi \qquad (17)$$

Consider the quadratic form $F(y,\xi)$ defined by

$$F(y,\xi) = y_1 \cdot \xi_1 = \Psi_t(x,t)\cdot\sigma(x,\theta)\Psi_t(x,t). \qquad (18)$$

The pair (A,B) is L^2- controllable since the matrix operator A is stable.

Consider the eigenvalue equation for A

$$Av = \lambda v. \qquad (19)$$

Denote by λ_i the eigenvalues of operator Λ and by e_k its eigenvectors, such that $\{e_k\}_k$ forms a basis of $L^2(0,1)$.

Vector v can be decomposed in this basis $\{e_k\}_k$ as

$$v_i = \sum_k c_i^k e_k\,.$$

Eigenvalue equation (19):

$$\begin{cases} -\upsilon_1 = \lambda\upsilon_2 \\ -\sigma_0\upsilon_1 + A_0\upsilon_2 = \lambda\upsilon_1 \\ -A_0\upsilon_3 = \lambda\upsilon_3 \end{cases}$$

$$\begin{cases} -\sum\limits_k c_1^k e_k = \lambda\sum\limits_k c_2^k e_k \\ -\sigma_0\sum\limits_k c_1^k e_k + \sum\limits_k \lambda_k c_2^k e_k = \lambda\sum\limits_k c_1^k e_k \\ -\sum\limits_k \lambda_k c_3^k e_k = \lambda\sum\limits_k c_3^k e_k \end{cases} \tag{20}$$

Verification of the frequency domain condition:

Functions $\Psi(x,t), \theta(x,t), \xi(x,t)$ can be decomposed by $\{e_k\}_k$ as follows:

$$\Psi(x,t) = \sum_k \Psi^k(t)e_k, \quad \theta(x,t) = \sum_k \theta^k(t)e_k,$$

$$\xi(x,t) = \sum_k \xi^k(t)e_k.$$

Introduce the quadratic form $(\Pi_0(i\omega)\xi,\xi) = \tilde{F}(y,\xi)$, where $\tilde{F}(y,\xi)$ is the extension of the quadratic form $F(y,\xi)$ to the Hermitian form $(\mathbf{F3})$.

Then the matrix-function $\Pi_0(i\omega)$ can be presented as

$$(\Pi_0(i\omega)\tilde{\xi},\tilde{\xi}) = \sum_k (\Pi_0^k(i\omega)\widetilde{\xi^k},\widetilde{\xi^k}). \tag{21}$$

Fourier transform with respect to t:

$$\begin{aligned} &-\omega^2\widetilde{\Psi^k}(i\omega) + i\omega\sigma_0\widetilde{\Psi^k}(i\omega) - \lambda_k\widetilde{\Psi^k}(i\omega) + \widetilde{\xi_1^k}(i\omega) = 0 \\ &i\omega\widetilde{\theta^k}(i\omega) + \lambda_k\widetilde{\theta^k}(i\omega) - \widetilde{\xi_2^k}(i\omega) = 0 \end{aligned} \tag{22}$$

From (22) $\widetilde{\Psi^k}$ and $\widetilde{\theta^k}$ can be expressed in terms of $\widetilde{\xi_1^k}, \widetilde{\xi_2^k}$ in the following way:

$$\widetilde{\Psi^k}(i\omega) = \chi_0(i\omega, \lambda_k)\xi_1^k(i\omega),$$
$$\widetilde{\theta^k}(i\omega) = \chi_1(i\omega, \lambda_k)\xi_2^k(i\omega),$$

where

$$\chi_0(i\omega, \lambda_k) = (\omega^2 - i\omega\sigma_0 + \lambda_k)^{-1},$$
$$\chi_1(i\omega, \lambda_k) = (i\omega + \lambda_k)^{-1}.$$

$(\Pi_0^k(i\omega)\widetilde{\xi^k}, \widetilde{\xi^k})$ from (21) can be written as follows:

$$(\Pi_0^k(i\omega)\widetilde{\xi^k}, \overline{\widetilde{\xi^k}})) = \mathrm{Re}\widetilde{\Psi_t^k}\,\overline{\widetilde{\xi_1^k}} = \mathrm{Re}(i\omega\chi_0)|\widetilde{\xi_1^k}(i\omega)|^2$$

Here the matrix $\Pi_0^k(i\omega)$ has the following form

$$\Pi_0^k(i\omega) = \begin{pmatrix} \mathrm{Re}(i\omega\chi_0) & 0 \\ 0 & 0 \end{pmatrix} \tag{23}$$

We have to check that

$$\mathrm{Re}(i\omega\chi_0) < 0\,, \forall\omega \in \mathbb{R}\,, \omega \neq 0 \tag{24}$$

Condition (24) is equivalent to $\mathrm{Re}(\frac{i\omega}{\omega^2 - i\omega\sigma_0 + \lambda_k}) < 0$.

This is satisfied if $-\omega^2\sigma_0 < 0, \forall\omega \neq 0$.

Remark We can also consider another quadratic form instead of (18)

$$F(y, \xi) = -y_3\xi_2 + ay_1\xi_1 + b\xi_1^2 \tag{25}$$

For the slightly modified version of equation (13):

$$\begin{cases} \Psi_{tt} + \sigma(x,\theta)\Psi_t - \Psi_{xx} = 0 \\ \theta_t - \theta_{xx} = \sigma(x,\theta)\Psi_t^2 + \varepsilon\Psi \end{cases} \tag{26}$$

6. Experimental results

Consider system (13) - (14) in the form

$$\begin{cases} h_t + \sigma(x,\theta)h - \Psi_{xx} = 0 \\ \Psi_t = h \\ \theta_t - \theta_{xx} = \sigma(x,\theta)h^2 \end{cases} \tag{27}$$

Initial-boundary conditions:

$$\begin{aligned} &\Psi(0,t) = \theta(0,t) = 0, \\ &\Psi(1,t) = \theta(1,t) = 0 \quad \forall t \in [0,T] \\ &\Psi(x,0) = \Psi_0(x), h(x,0) = h_0(x), \theta(x,0) = \theta_0(x), \forall x \in \Omega \end{aligned} \tag{28}$$

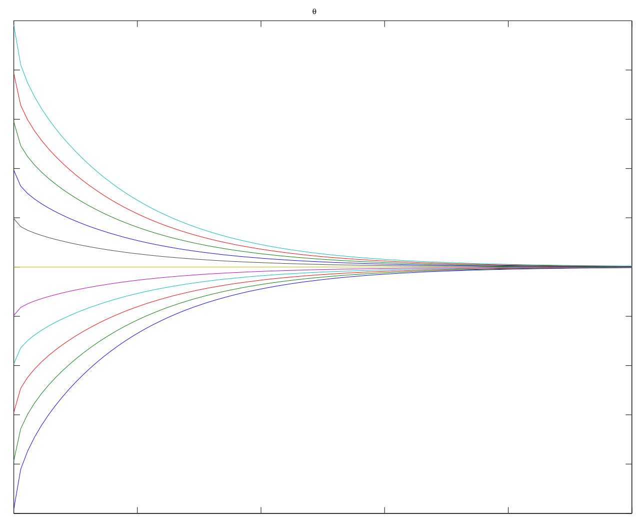
$h_0(x) = p \cdot (1 - |2x-1|)$, $\Psi_0(x) \equiv 0$, $\theta_0(x) = p \cdot (1 - |2x-1|)$, where $p \in \mathbb{R}$ is some parameter.

J. Morgan, H.-M. Yin, 2001

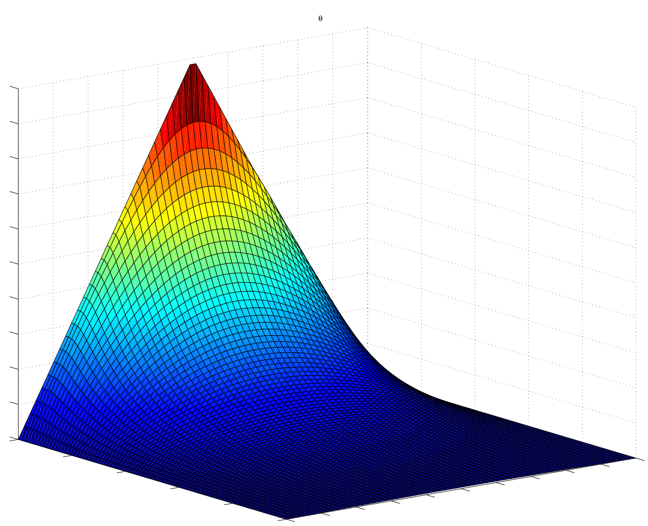
Electrical conductivity: $\sigma(x,\theta) = c + \theta(x,t)$, where c is some positive constant.

For convention: Denote $h(x,t)$ by $\Psi_t(x,t)$.

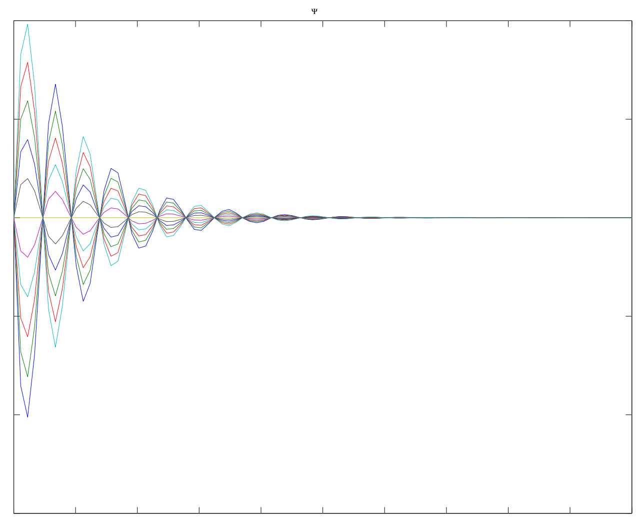
Consider solutions $(\Psi_t^p(x,t), \Psi^p(x,t), \theta^p(x,t))$ with $p \in [-0.5, 0.5]$.



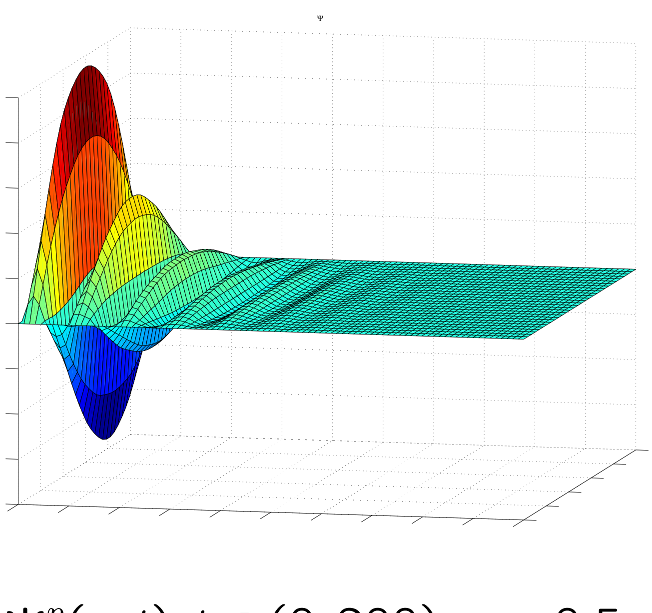
$\theta^p(x_0, t), x_0 = 0.5, t \in (0, 0.25)$



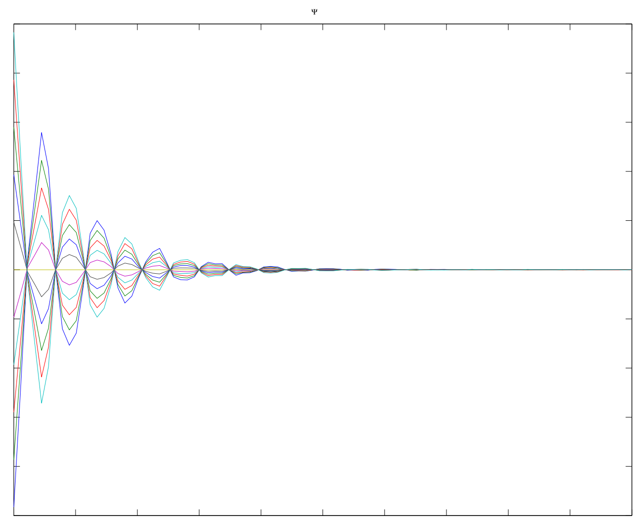
$\theta^p(x, t), t \in (0, 0.25), p = 0.5$



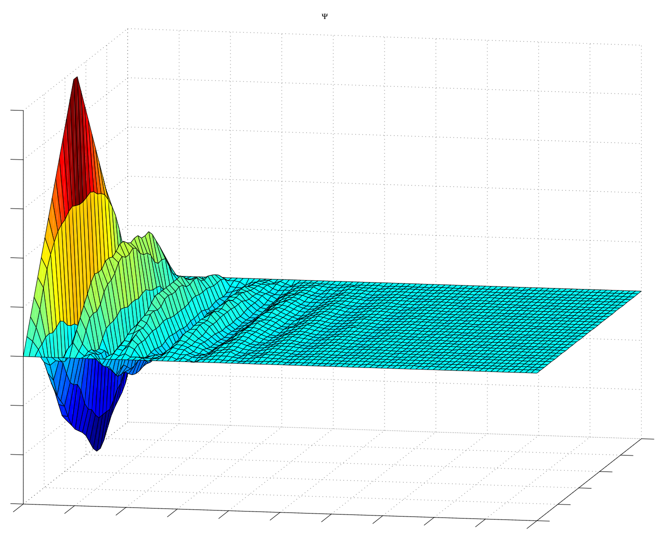
$\Psi^p(x_0, t), x_0 = 0.5, t \in (0, 200)$



$\Psi^p(x, t), t \in (0, 200), p = 0.5$



$\Psi_t^p(x_0, t), x_0 = 0.5, t \in (0, 200)$



$\Psi_t^p(x, t), t \in (0, 200), p = 0.5$

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