Frequency-domain conditions for the existence of almost-periodic solutions in coupled PDEs

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1. Evolutionary variational systems

Suppose: Y_0 is a real Hilbert space with $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ as scalar product resp. norm.

$A : \mathcal{D}(A) \subset Y_0$ is a closed (unbounded) densely defined linear operator. Y_1 is defined as $\mathcal{D}(A)$ equipped with the scalar product

$$(y,\eta)_1 := ((\beta I - A)y,\, (\beta I - A)\eta)_0\,, \quad y,\eta \in \mathcal{D}(A)\,, \tag{1}$$

where $\beta \in \rho(A)$ ($\rho(A)$ is the resolvent set of A)

Y_{-1} is the completion of Y_0 with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|_0$. Thus we have the dense and continuous imbedding

$$Y_1 \subset Y_0 \subset Y_{-1} \tag{2}$$

(Hilbert space rigging structure). The duality pairing $(\cdot,\cdot)_{-1,1}$ on $Y_1 \times Y_{-1}$ is the unique extension by continuity of the functionals $(\cdot, y)_0$ with $y \in Y_1$ onto Y_{-1}.

If $-\infty \le T_1 < T_2 \le +\infty$ are arbitrary numbers, we define the norm for Bochner measurable functions in $L^2(T_1,T_2;Y_j)\,,\ j = 1,0,-1$, through

$$\|y\|_{2,j} := (\int_{T_1}^{T_2} \|y(t)\|_j^2\, dt)^{1/2}\,. \tag{3}$$

For an arbitrary interval J in $\mathbb{R}$ denote by $\mathcal{W}(J)$ the space of functions $y(\cdot) \in L^2_{\text{loc}}(J;Y_1)$ for which $\dot y\,(\cdot) \in L^2_{\text{loc}}(J;Y_{-1})$ equipped with the norm defined for any compact interval $[T_1,T_2]$ by

$$\|y\,(\cdot)\|_{\mathcal{W}(T_1,T_2)} := (\|y\,(\cdot)\|^2_{2,1} + \|\dot y\,(\cdot)\|^2_{2,-1})^{1/2}\,. \tag{4}$$

Assume also: Any function from $\mathcal{W}(J)$ belongs to $C(J;Y_0)$.

Ξ is an other real Hilbert space with scalar product $(\cdot,\cdot)_\Xi$ and norm $\|\cdot\|_\Xi$,

$J \subset \mathbb{R}$ is an arbitrary interval.

Introduce

$$A : Y_1 \to Y_{-1} \quad \text{and} \quad B : \Xi \to Y_{-1} \tag{5}$$

and the maps

$$\varphi : J \times Y_1 \to \Xi\,, \tag{6}$$

and
$$f : J \to Y_{-1}\,. \tag{7}$$

Consider for a.a. $t \in J$ the evolutionary variational equation

$$(\dot{y}(t) - Ay(t) - B\varphi(t, y(t)) - f(t)\,,\ \eta - y(t))_{-1,1} = 0\,, \qquad \forall\, \eta \in Y_1\,. \tag{8}$$

For any $f \in L^2_{\text{loc}}(J; Y_{-1})$ a function $y(\cdot) \in \mathcal{W}(J) \cap C(J; Y_0)$ is said to be a solution of (8) if this equality is satisfied for all test functions $\eta \in Y_1$.

2. Further assumptions

(A1) For any $t \in J$ the map $\mathcal{A}(t)y := -Ay - B\,\varphi\,(t, y)\ :\ Y_1 \to Y_{-1}$ is semicontinuous, i.e., for any $t \in J$ and any $y, \eta, z \in Y_1$ the $\mathbb{R}$-valued function $\tau \mapsto (\mathcal{A}(t)(y - \tau\eta), z)_{-1,1}$ is continuous.

(A2) For any $\eta \in Y_1$ and any bounded set $S \subset Y_1$ the family of functions $\{(B\varphi\,(\cdot, y), \eta)_{-1,1}\,, y \in S\}$ is equicontinuous on any compact subinterval of J.

(A3) $\varphi\,(\cdot, 0) \equiv 0$ on J and there exist operators $N \in \mathcal{L}(Y_1, \Xi)$ and $M = M^* \in \mathcal{L}(\Xi, \Xi)$ such that

$$\begin{aligned} &(\varphi\,(t, y_1) - \varphi\,(t, y_2), N(y_1, -y_2))_{\Xi} \\ &\geq\ (\varphi\,(t, y_1) - \varphi\,(t, y_2),\ M\big(\varphi\,(t, y_1) - \varphi\,(t, y_2)\big)_{\Xi}, \\ &\forall\ t \in J,\ \forall\ y_1, y_2 \in Y_1\,. \end{aligned} \tag{9}$$

(A4) There exists a quadratic form $\mathcal{G}$ on $Y_0 \times \Xi$ and a continuous functional $\Phi : Y_0 \to \mathbb{R}_+$ such that for any $y_1(\cdot), y_2(\cdot) \in L^2_{\text{loc}}(J; Y_0)$ and a.a. $s, t \in J, s < t$, we have

$$\int_s^t \mathcal{G}(y_1(\tau) - y_2(\tau), \varphi(\tau, y_1(\tau)) - \varphi(\tau, y_2(\tau)))d\tau \\ \geq \frac{1}{2}\Phi(y_1(\tau) - y_2(\tau))\big|_s^t . \qquad (10)$$

Furthermore, there are two constants $0 < \rho_1 < \rho_2$ such that

$$\rho_1 \|y\|_0^2 \leq \Phi(y) \leq \rho_2 \|y\|_0^2 , \quad \forall\, y \in Y_0. \qquad (11)$$

Suppose that there exists a number $\lambda > 0$ such that the following assumptions are satisfied:

(A5) For any $T > 0$ and any $f \in L^2(0, T; Y_{-1})$ the problem

$$\dot{y} = (A + \lambda I)\, y + f(t), y(0) = y_0, \qquad (12)$$

is well-posed, i.e., for arbitrary $y_0 \in Y_0, f(\cdot) \in L^2(0, T; Y_{-1})$ there exists a unique solution $y(\cdot) \in \mathcal{W}(0, T)$ with $\dot{y}\,(\cdot) \in L^2(0, T; Y_{-1})$ satisfying the equation in a variational sense and depending continuously on the initial data, i.e.,

$$\|y(\cdot)\|^2_{\mathcal{W}(0,T)} \leq c_1 \|y_0\|_0^2 + c_2 \|f(\cdot)\|^2_{2,-1} , \qquad (13)$$

where $c_1 > 0$ and $c_2 > 0$ are some constants. Furthermore it is supposed that any solution of $\dot{y} = (A + \lambda I)\, y\,,\ y\,(0) = y_0$, is exponentially decreasing for $t \to +\infty$, i.e., there exist constants $c_3 > 0$ and $\varepsilon > 0$ such that

$$\|y\,(t)\|_0 \leq c_3\, e^{-\varepsilon t} \|y_0\|_0\ ,\ t > 0\ . \qquad (14)$$

(A6) The operator $A + \lambda I \in \mathcal{L}(Y_1, Y_{-1})$ is regular, i.e., for any $T > 0, y_0 \in Y_1, z_T \in Y_1$ and $f \in L^2(0, T; Y_0)$ the solution of the direct problem

$$\dot{y} = (A + \lambda I)y + f(t), \quad y\,(0) = y_0 \qquad (15)$$

and of the dual problem

$$\dot{z} = -(A + \lambda I)^* z + f(t), \quad z(0) = z_T \tag{16}$$

are strongly continuous in t in the norm of Y_1.

(A7) The pair $(A + \lambda I, B)$ is L^2-controllable, i.e., for arbitrary $y_0 \in Y_0$ there exists a control $\xi(\cdot) \in L^2(0, +\infty; \Xi)$ such that the problem $\dot{y} = (A + \lambda I)y + B\xi, y(0) = y_0$, is well-posed in the variational sense on $(0, +\infty)$.

(A8) Let denote by H^c and L^c the complexification of a linear space H and a linear operator L, respectively, by
$\chi(s) = (sI^c - A^c)^{-1} B^c$, $s \notin \rho(A^c)$, the transfer operator, and by $\mathcal{G}^c$ the Hermitian extension of $\mathcal{G}$.

There exist a number $\Theta > 0$ such that with ρ_2 from (11) and the imbedding constant γ from $Y_1 \subset Y_0$

$$\begin{aligned} &\Theta\big[\mathrm{Re}\big(\xi, N^c \chi(i\omega - \lambda)\,\xi\big)_{\Xi^c} + (\xi, M^c \xi)_{\Xi^c}\big] \\ &+ \mathcal{G}^c\left(\chi\left(i\omega - \lambda\right)\xi, \xi\right) + \gamma\lambda\rho_2 \|\chi\left(i\omega - \lambda\right)\xi\|^2_{Y_1^c} < 0\,, \\ &\forall\, \omega \in \mathbb{R},\ \forall\, \xi \in \Xi^c\,. \end{aligned} \tag{17}$$

(A9) For any $y_0 \in Y_0$ there exist at least one solution $y(\cdot)$ of (8) on $\mathbb{R}_+$ with $y(0) = y_0$.

Uniqueness to the right and the continuous dependence of solutions on initial states:

a) If y_1, y_2 are two solutions of (8) on $\mathbb{R}_+$ and $y_1(t_0) = y_2(t_0)$ for some $t_0 \geq 0$ then $y_1(t) = y_2(t)\,,\ \forall\, t \geq t_0$.

b) If $y(\cdot, a_k),\ k = 1, 2, \ldots$, are solutions of (8) with $y(t_0, a_k) = a_k$ on $J_0 = [t_0, t_1]$ or $J_0 = [t_1, t_0]$ and $a_k \to a$ for $k \to \infty$ in Y_0 then there exists a subsequence $k_n \to \infty$ with $y(\cdot, a_{k_n}) \to y$ for $n \to \infty$ in $C(J_0; Y_0)$ and y is a solution of (8) on J_0 with $y(t_0) = a$.

3. Existence of bounded solutions

Let $(E, \|\cdot\|_E)$ be a Banach space.

Denote by $C_b(\mathbb{R}; E) \subset C(\mathbb{R}; E)$ the subspace of bounded continuous functions with the norm $\|f\|_{C_b} = \sup_{t \in \mathbb{R}} \|f(t)\|_E$.

The space $BS^2(\mathbb{R}; E)$ of *bounded* (with exponent 2) *in the sense of Stepanov functions* is the subspace of all functions f from $L^2_{\text{loc}}(\mathbb{R}; E)$ which have a finite norm

$$\|f\|^2_{S^2} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(\tau)\|^2_E \, d\tau \ . \tag{18}$$

Theorem 3.1 Suppose that the assumptions **(A3)** – **(A9)** are satisfied and

$$f \in BS^2(\mathbb{R}_+; Y_{-1}) \ . \tag{19}$$

Then any solution $y(\cdot)$ of (8) belongs to $C_b\,(\mathbb{R}_+; Y_0)$.

4. Existence of almost periodic solutions

Let $f : \mathbb{R} \to E$ be continuous. If $\varepsilon > 0$, then a number $T \in \mathbb{R}$ is called ε-*almost period* of f if $\sup_{t \in \mathbb{R}} \|f\,(t+T) - f\,(t)\|_E \leq \varepsilon$.

The function f is called *Bohr almost periodic* or *uniformly almost periodic* (shortly $f \in \text{CAP}(\mathbb{R}; E)$ or uniformly a.p.) if for each $\varepsilon > 0$ there is $R > 0$ such that each interval $(r, r+R) \subset \mathbb{R}$ $(r \in \mathbb{R})$ contains at least one ε-almost period of f.

For a function $f \in L^2_{\text{loc}}(\mathbb{R}; E)$ define the *Bochner transform* f^b by

$$f^b(t) := f(t+\eta) \ , \ \eta \in [0,1] \ , \ t \in \mathbb{R} \ ,$$

as a (continuous) function with values in $L^2(0,1; E)$.

A function $f \in BS^2(\mathbb{R}; E)$ is called an *almost periodic function in the sense of Stepanov* (shortly S^2-a.p.) if $f^b \in$ CAP $(\mathbb{R}; L^2(0,1;E))$.

The space of S^2-a.p. functions with values in E is denoted by $S^2(\mathbb{R}; E)$. Obviously, CAP$(\mathbb{R}; E) \subset S^2(\mathbb{R}; E)$.

(A10) The family of functions $\{\varphi(\cdot, y), y \in Y_1\}$ is uniformly almost periodic on any set $\{y \in Y_1 : \|y\|_1 \le \text{const}\}$.

Theorem 4.1 Under the assumptions **(A3)** – **(A9)** there exists for any $f \in BS^2(\mathbb{R}; Y_{-1})$ a unique bounded on $\mathbb{R}$ solution $y_*(\cdot)$ of (8). This solution is exponentially stable in the whole, i.e., there exist positive constants $c > 0$ and $\varepsilon > 0$ such that for any other solution y of (8), any $t_0 \in \mathbb{R}$ and any $t \ge t_0$ we have

$$\|y(t) - y_*(t)\|_0 \le c\, e^{-\varepsilon(t-t_0)} \|y(t_0) - y_*(t_0)\|_0 \,. \qquad (20)$$

If φ satisfies **(A10)** and $f \in S^2(\mathbb{R}; Y_{-1})$ then $y_*(\cdot)$ belongs to CAP $(\mathbb{R}; Y_0)$.

5. Examples

Example 5.1

$$Y_0 = L^2(0,1), \quad Y_1 = W^{1,2}(0,1)$$
$$(u,v)_1 = \int_0^1 (uv + u_x v_x)\, dx \qquad (21)$$

$$A : Y_1 \to Y_{-1}\,, (Au, v)_{-1,1} = \int_0^1 (Au)(x)v(x)dx := - \int_0^1 (a u_x v_x + b u v)\, dx\ , \forall\, u, v \in W^{1,2}(0,1) \qquad (22)$$

$$\begin{aligned} &('' Au = au - bu_x'') \\ &\Xi = \mathbb{R}\,,\ B : \Xi \to Y_{-1}\,, \\ &(B\,\xi, v)_{-1,1} := a\,\xi v(1)\,, \quad \forall\, \xi \in \mathbb{R},\ \forall\, v \in W^{1,2}(0,1) \end{aligned} \qquad (23)$$

$$('' B = a\delta(x-1)'')$$
$$u_x(0,t) = 0\,, \quad u_x(1,t) = g(w(t)) + f(t)\,, \qquad (24)$$

$g : \mathbb{R} \to \mathbb{R}$ continuous, $f \in L^2_{\text{loc}}(\mathbb{R}) \cap$ CAP $(\mathbb{R})$

$K : Y_1 \to \mathbb{R}$ linear continuous, $K(u) = \int_0^1 k(x)u(x,t)\,dx$,

$\varphi : L^2(0,1) \to \mathbb{R}$ given by

$$u \in L^2(0,1) \mapsto w(\cdot) = K\left(u(\cdot)\right) \mapsto g(w(\cdot)) \in \mathbb{R} \qquad (25)$$

$$\exists\, \mu_0 > 0 \quad \forall\, w_1, w_2 : \; 0 \le (g(w_1) - g(w_2))(w_1 - w_2) \le \mu_0 (w_1 - w_2)^2 , \qquad (26)$$

$$\exists\, c_1 > 0 \quad \forall\, w_1, w_2 \in \mathcal{W}(0,T) \quad \forall\, s < t,\; s,t \in (0,T) :$$
$$\int_s^t (\dot{w}_1 - \dot{w}_2)\,(\varphi(w_1) - \varphi(w_2))\,d\tau \ge c_1 |w_1(\tau) - w_2(\tau)|^2\big|_s^t \qquad (27)$$

$$\chi(s) = K(\tilde{u}(x,s)),\; s \in \mathbb{C} ,$$
$$s\tilde{u} = a\tilde{u}_{xx} - b\tilde{u},\; \tilde{u}_x(0,t) = 0,\; \tilde{u}_x(1,t) = 0 \qquad (28)$$

$$\chi(s) = K\left(\frac{ab \cosh(\frac{1}{a}\sqrt{s+b}x)}{\sqrt{s+b}\sinh(\frac{1}{a}\sqrt{s+b}} \right) \qquad (29)$$

$$\exists\, \Theta > 0 \quad \exists\, \varepsilon > 0 \quad \exists\, \lambda > 0 \quad \forall \omega \in \mathbb{R} :$$
$$\mu_0 \operatorname{Re}\chi(i\omega - \lambda) + \Theta \operatorname{Re}\left(i\omega\chi(i\omega - \alpha)\right) \ge \varepsilon , \qquad (30)$$

$$\exists\, m > 0 \quad \forall u \in W^{1,2}(0,1) \;:\; K(u) \ge m\|u\|_1^2 \qquad (31)$$

$\Rightarrow$ assumptions of Theorem 4.1 are satisfied

Example 5.2

Consider the coupled system of Maxwell's equation and heat transfer equation

$$\begin{cases} \Psi_{tt} + \sigma(x,\theta)\Psi_t - \Psi_{xx} = 0 \\ \theta_t - \theta_{xx} = \sigma(x,\theta)\Psi_t^2 \end{cases} \tag{32}$$

Initial-boundary conditions:

$$\begin{array}{l} \Psi(0,t) = g_1(t) = \cos(t) + \cos(\sqrt[2]{t})\,, \quad \forall t \in [0,T] \\ \theta(0,t) = \theta(1,t) = 0\,, \quad \forall t \in [0,T] \\ \Psi(1,t) = g_2(t) = 0\,, \quad \forall t \in [0,T] \\ \Psi(x,0) = \Psi_0(x) = 2 - 2x\,, \quad \forall x \in \Omega \\ \Psi_t(x,0) = \Psi_1(x)\,, \quad \forall x \in \Omega \\ \theta(x,0) = \theta_0(x), \quad \forall x \in \Omega \end{array} \tag{33}$$

Here $\Omega = (0,1)$.

Energy inequality:

$$\sup_{0\le t\le T} \int_0^1 [\Psi_t^2 + \Psi_x^2]dx + \int_0^T \int_0^1 \sigma(x,t,\theta)\Psi_t^2 dxdt$$
$$\le C_1 + C_2 \int_0^T \int_0^1 |\theta| dxdt\,,$$

where the constants C_1 and C_2 depend only on known data.

System in terms of operator equations in some function spaces:

$$y(x,t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \Psi_t(x,t) \\ \Psi(x,t) \\ \theta(x,t) \end{pmatrix}, \tag{34}$$

$$\xi(x,t) = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \sigma(x,\theta)\Psi_t(x,t) \\ \sigma(x,\theta)\Psi_t^2(x,t) \end{pmatrix}.$$

Let us define operators A, B from equation (8). Let Λ be the self-adjoint positiv operator, generated on $L^2(0,1)$ by the differential expression $\Lambda(v) = -v_{xx}$ and zero boundary conditions (33).

Consider the following spaces $Y_0 = L^2(0,1)\times L^2(0,1)\times L^2(0,1)$, $Y_1 = W^{1,2}(0,1)\times W^{1,2}(0,1)\times W^{1,2}(0,1)$ and $\Xi = L^2(0,1)\times L^2(0,1)$ as defined in Section 1.

Then operators A and B are defined as follows:

$$A = \begin{bmatrix} -\sigma_0 I & \Lambda & 0 \\ -I & 0 & 0 \\ 0 & 0 & -\Lambda \end{bmatrix}, B = \begin{bmatrix} -I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}, \tag{35}$$

Here the constant $\sigma_0 > 0$ is derived from decomposition $\sigma(x,\theta) = \sigma_0 + \bar{\sigma}(x,\theta)$.

Finally, system (32) can be written in terms of the operator equation

$$\frac{dy}{dt} = Ay + B\xi \tag{36}$$

Consider the quadratic form $F(y,\xi)$ defined by

$$F(y,\xi) = y_1 \cdot \xi_1 = \Psi_t(x,t)\cdot\sigma(x,\theta)\Psi_t(x,t). \tag{37}$$

The pair (A,B) is L^2- controllable since the matrix operator A is stable.

Suppose that $\{e_k\}_k$ forms a basis of $L^2(0,1)$.

Verification of the frequency domain condition:

Functions $\Psi(x,t), \theta(x,t), \xi(x,t)$ can be decomposed by $\{e_k\}_k$ as follows:

$$\Psi(x,t) = \sum_k \Psi^k(t)e_k, \quad \theta(x,t) = \sum_k \theta^k(t)e_k, \tag{38}$$

$$\xi(x,t) = \sum_k \xi^k(t) e_k. \tag{39}$$

Introduce the quadratic form $(\Pi_0(i\omega)\xi, \xi) = \tilde{F}(y,\xi)$, where $\tilde{F}(y,\xi)$ is the extension of the quadratic form $F(y,\xi)$ to the Hermitian form $(\mathbf{A3})$.

Then the matrix-function $\Pi_0(i\omega)$ can be presented as

$$(\Pi_0(i\omega)\tilde{\xi}, \tilde{\xi}) = \sum_k (\Pi_0^k(i\omega)\widetilde{\xi^k}, \widetilde{\xi^k})\,. \tag{40}$$

Fourier transform with respect to t:

$$\begin{array}{l} -\omega^2 \widetilde{\Psi^k}(i\omega) + i\omega\sigma_0 \widetilde{\Psi^k}(i\omega) - \lambda_k \widetilde{\Psi^k}(i\omega) + \widetilde{\xi_1^k}(i\omega) = 0 \\ i\omega\widetilde{\theta^k}(i\omega) + \lambda_k \widetilde{\theta^k}(i\omega) - \widetilde{\xi_2^k}(i\omega) = 0 \end{array} \tag{41}$$

From (41) $\widetilde{\Psi^k}$ and $\widetilde{\theta^k}$ can be expressed in terms of $\widetilde{\xi_1^k}, \widetilde{\xi_2^k}$ in the following way:

$$\begin{array}{l} \widetilde{\Psi^k}(i\omega) = \chi_0(i\omega, \lambda_k)\xi_1^k(i\omega), \\ \widetilde{\theta^k}(i\omega) = \chi_1(i\omega, \lambda_k)\xi_2^k(i\omega), \end{array}$$

where

$$\begin{array}{l} \chi_0(i\omega, \lambda_k) = (\omega^2 - i\omega\sigma_0 + \lambda_k)^{-1}, \\ \chi_1(i\omega, \lambda_k) = (i\omega + \lambda_k)^{-1}. \end{array}$$

$(\Pi_0^k(i\omega)\widetilde{\xi^k}, \widetilde{\xi^k})$ from (40) can be written as follows:

$$(\Pi_0^k(i\omega)\widetilde{\xi^k}, \overline{\widetilde{\xi^k}})) = \mathrm{Re}\widetilde{\Psi_t^k}\ \overline{\widetilde{\xi_1^k}} = \mathrm{Re}(i\omega\chi_0)|\widetilde{\xi_1^k}(i\omega)|^2$$

Here the matrix $\Pi_0^k(i\omega)$ has the following form

$$\Pi_0^k(i\omega) = \begin{pmatrix} \mathrm{Re}(i\omega\chi_0) & 0 \\ 0 & 0 \end{pmatrix} \tag{42}$$

We have to check that

$$\text{Re}(i\omega\chi_0) < 0\,, \forall\omega \in \mathbb{R}\,, \omega \neq 0 \qquad (43)$$

Condition (43) is equivalent to $\text{Re}(\frac{i\omega}{\omega^2 - i\omega\sigma_0 + \lambda_k}) < 0$.

This is satisfied if $-\omega^2\sigma_0 < 0, \forall\omega \neq 0$.

6. Numerical results

Consider system (32) - (33) in the form

$$\begin{cases} h_t + \sigma(x,\theta)h - \Psi_{xx} = 0 \\ \Psi_t = h \\ \theta_t - \theta_{xx} = \sigma(x,\theta)h^2 \end{cases} \qquad (44)$$

(**S**) Initial-boundary conditions (without perturbations):

$$\begin{array}{l} \Psi(0,t) = \theta(0,t) = 0, \\ \Psi(1,t) = \theta(1,t) = 0 \quad \forall t \in [0,T] \\ \Psi(x,0) = \Psi_0(x), h(x,0) = h_0(x), \theta(x,0) = \theta_0(x), \forall x \in \Omega \end{array} \qquad (45)$$

$h_0(x) = p \cdot (1 - |2x-1|)$, $\Psi_0(x) \equiv 0$, $\theta_0(x) = p \cdot (1 - |2x-1|)$, where $p \in \mathbb{R}$ is some parameter.

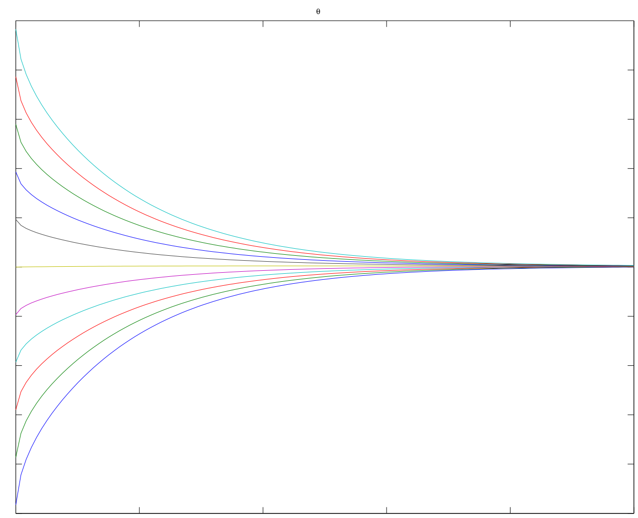
J. Morgan, H.-M. Yin, 2001

Electrical conductivity: $\sigma(x,\theta) = c + \theta(x,t)$, where c is some positive constant.

For convention: Denote $h(x,t)$ by $\Psi_t(x,t)$.

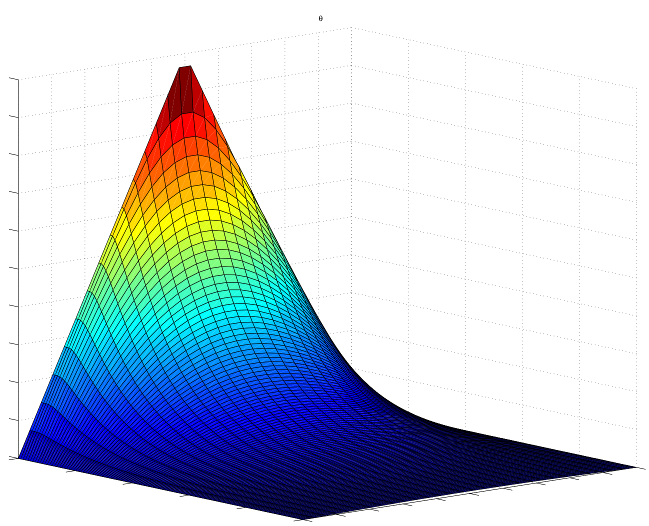
Consider solutions $(\Psi_t^p(x,t), \Psi^p(x,t), \theta^p(x,t))$ with $p \in [-0.5, 0.5]$.

(**S1**)



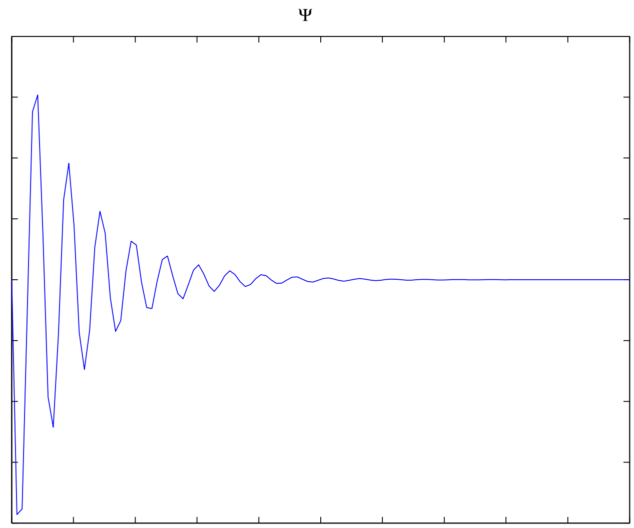
$\theta^p(x_0, t), x_0 = 0.5, t \in (0, 0.25)$

(**S2**)



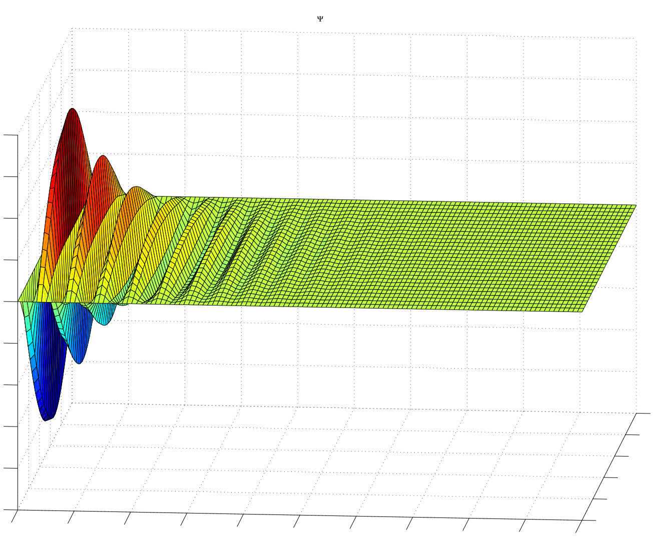
$\theta^p(x, t), t \in (0, 0.25), p = 0.5$

(**S3**)



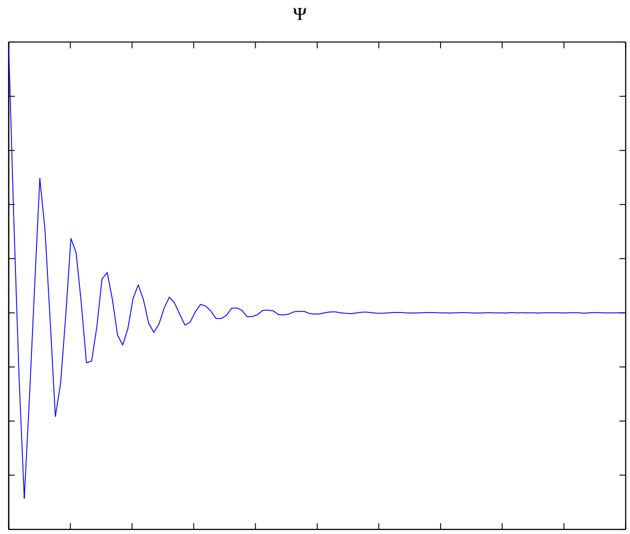
$\Psi^p(x_0, t), x_0 = 0.5, t \in (0, 200)$

(**S4**)



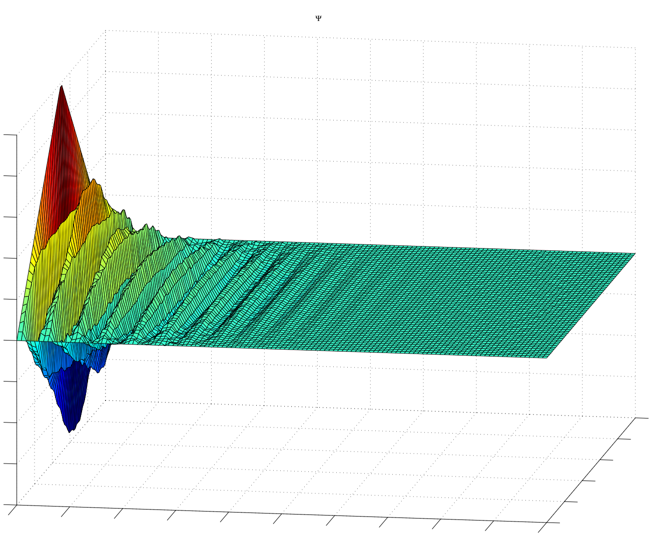
$\Psi^p(x, t), t \in (0, 200), p = 0.5$

(**S5**)



$\Psi_t^p(x_0, t), x_0 = 0.5, t \in (0, 200)$

(**S6**)

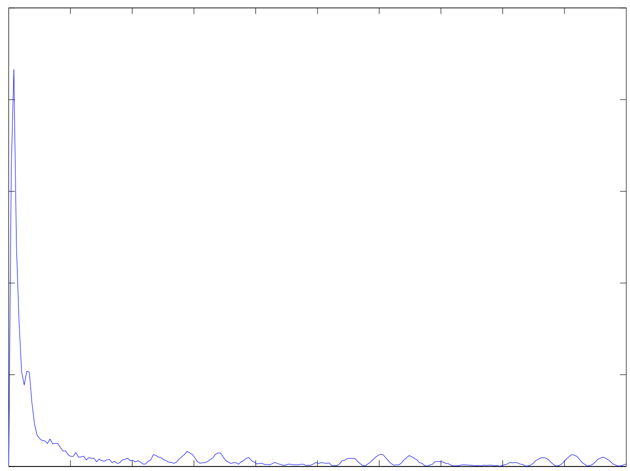


$\Psi_t^p(x, t), t \in (0, 200), p = 0.5$

(**P**) Initial-boundary conditions (with almost-periodic perturbations):

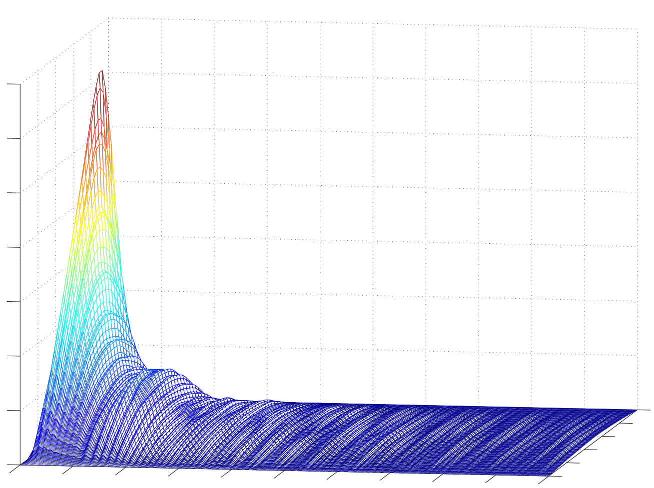
$$\Psi(x,0) = 2 - 2x; h(x,0) = 0; \theta(x,0) = 0;$$
$$\Psi(0,t) = g_1(t) = \cos(t) + \cos(\sqrt{2}t);$$
$$\Psi(1,t) = g_2(t) = 0; \theta(0,t) = \theta(1,t) = 0;$$

(**P1**)



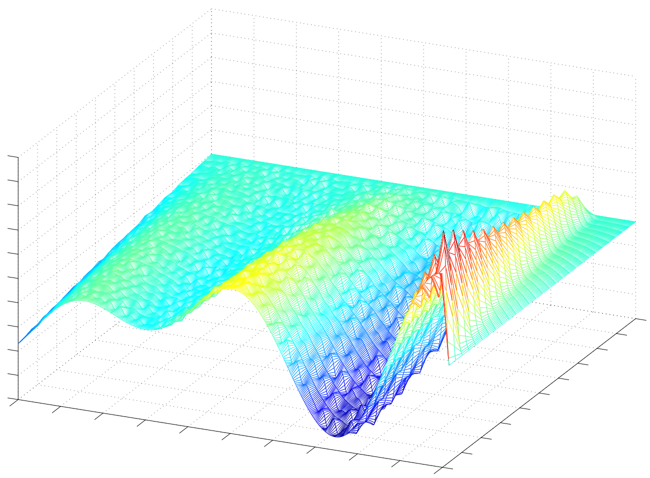
$\theta(x,t), \theta_0 = 0, t \in (0,50).$

(**P2**)



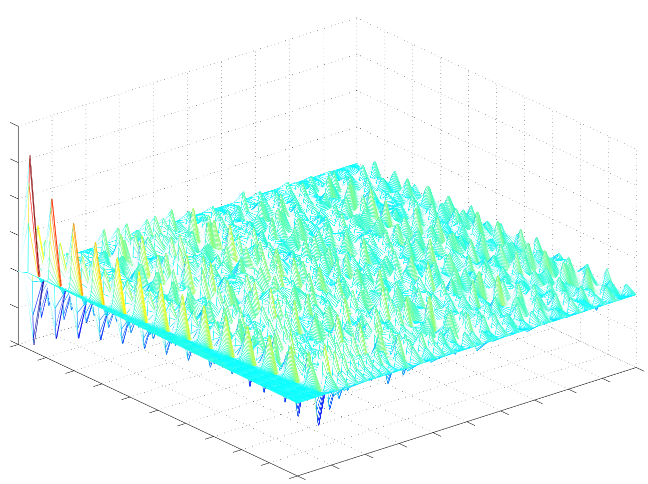
$\theta(x,t), \theta_0 = 0, t \in (0,10).$

(**P3**)



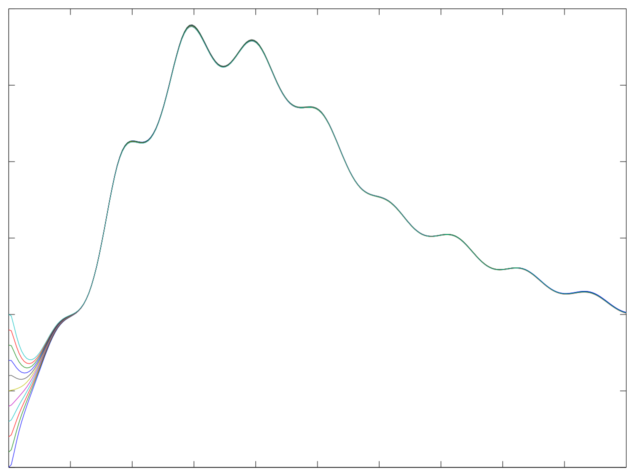
$\Psi(x,t), \Psi_0 = 1, t \in (0,10).$

(**P4**)



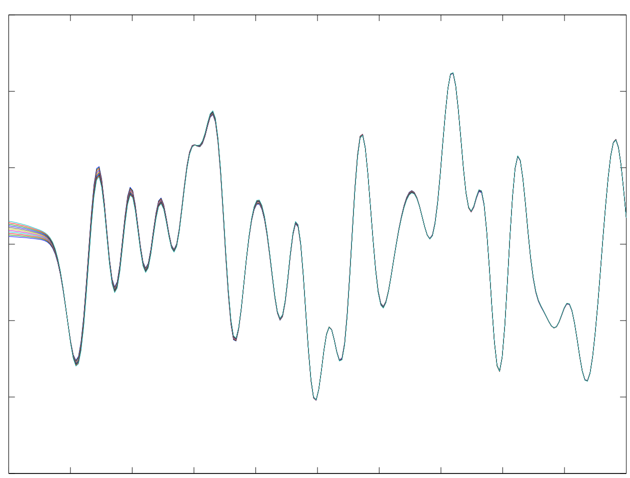
$h(x,t), h_0 = 0, t \in (0,10).$

(**P5**)



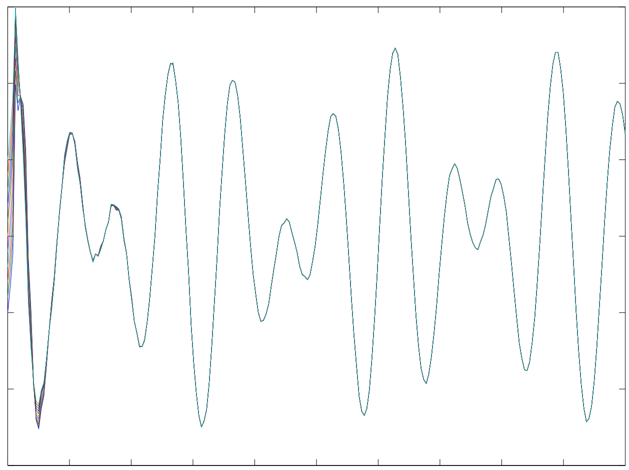
$\theta(x,t), \theta_0 \in (-0.5, 0.5), t \in (0,1).$

(**P6**)



$\Psi(x,t), \Psi_0 \in (1-0.5, 1+0.5), t \in (0,5).$

(**P7**)



$$h(x,t), h_0 \in (-0.5, 0.5), t \in (0, 50).$$

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