Absolute observation stability for evolutionary variational inequalities

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1. Introduction

Suppose: Y_0 a real Hilbert space, $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ the scalar product resp. the norm on Y_0,
$A : \mathcal{D}(A) \to Y_0$ the generator of a C_0-semigroup on Y_0, $Y_1 := \mathcal{D}(A)$.
For fixed $\beta \in \rho(A) \cap \mathbb{R}$ for any $y, \eta \in Y_1$ define

$$(y,\eta)_1 := ((\beta I - A)y,\, (\beta I - A)\eta)_0 \,. \tag{1}$$

Y_{-1} is the completion of Y_0 with respect to the norm,
$\|y\|_{-1} := \|(\beta I - A)^{-1} y\|_0$ is the scalar product

$$(y,\eta)_{-1} := \left((\beta I - A)^{-1} y,\ (\beta I - A)^{-1}\eta\right)_0, \qquad \forall\, y, \eta \in Y_{-1}. \tag{2}$$

$Y_1 \subset Y_0 \subset Y_{-1}$ is a continuous embedding, i.e., for $\alpha = 1, 0$,
$Y_\alpha \subset Y_{\alpha-1}$, $\|y\|_{\alpha-1} \le c\|y\|_\alpha$, $\forall\, y \in Y_\alpha$.

(Y_1, Y_0, Y_{-1}) is called a *Gelfand triple*.
For any $y \in Y_0$ and $z \in Y_1$ we have

$$|(y,z)_0| = |(\beta I - A)^{-1} y, ((\beta I - A)z)_0| \le \|y\|_{-1}\|z\|_1 \,. \tag{3}$$

Extend $(\cdot, z)_0$ by continuity onto Y_{-1}

$$|(y,z)_0| \le \|y\|_{-1}\|z\|_1 \,, \quad \forall\, y \in Y_{-1}, \forall\, z \in Y_1.$$

Denote this extension also by $(\cdot,\cdot)_{-1,1}$.
Consider the Bochner measurable functions in

$L^2(0,T;Y_j) \quad (j = 1, 0, -1)$

$$\|y(\cdot)\|_{2,j} := \left(\int_0^T \|y(t)\|_j^2\, dt\right)^{1/2}. \tag{4}$$

$\mathcal{L}_T$ is the space of functions $y \in L^2(0,T;Y_1)$, s.th. $\dot y \in L^2(0,T;Y_{-1})$.
$\mathcal{L}_T$ is equipped with the norm

$$\|y\|_{\mathcal{L}_T} := \left(\|y(\cdot)\|_{2,1}^2 + \|\dot y(\cdot)\|_{2,-1}^2\right)^{1/2}. \tag{5}$$

2. Evolutionary variational inequalities

Take $T > 0$ arbitrary and consider for a.a. $t \in [0,T]$ the evolutionary variational inequality

$$(\dot{y} - Ay - B\xi - f(t), \eta - y)_{-1,1} + \psi(\eta) - \psi(y) \geq 0\,, \qquad \forall\, \eta \in Y_1 \tag{6}$$

$$y(0) = y_0 \in Y_0\,,$$

$$w(t) = Cy(t)\,, \quad \xi(t) \in \varphi(t, w(t))\,, \tag{7}$$

$$\xi(0) = \xi_0 \in \mathcal{E}(y_0)\,,$$

$$z(t) = Dy(t) + E\,\xi(t)\,. \tag{8}$$

$C \in \mathcal{L}(Y_{-1}, W), D \in \mathcal{L}(Y_1, Z)$ and $E \in \mathcal{L}(\Xi, Z)$, Ξ, W and Z are real Hilbert spaces, $Y_1 \subset Y_0 \subset Y_{-1}$ is a real Gelfand triple and $A \in \mathcal{L}(Y_0, Y_{-1})$, $B \in \mathcal{L}(\Xi, Y_{-1}), \varphi : \mathbb{R}_+ \times W \to 2^{\Xi}$ is a set-valued map, $\psi : Y_1 \to \mathbb{R}_+$ and $f : \mathbb{R}_+ \to Y_{-1}$ are nonlinear maps.

Denote by $\|\cdot\|_{\Xi}$, $\|\cdot\|_W$, $\|\cdot\|_Z$ the norm in Ξ, W resp. Z .

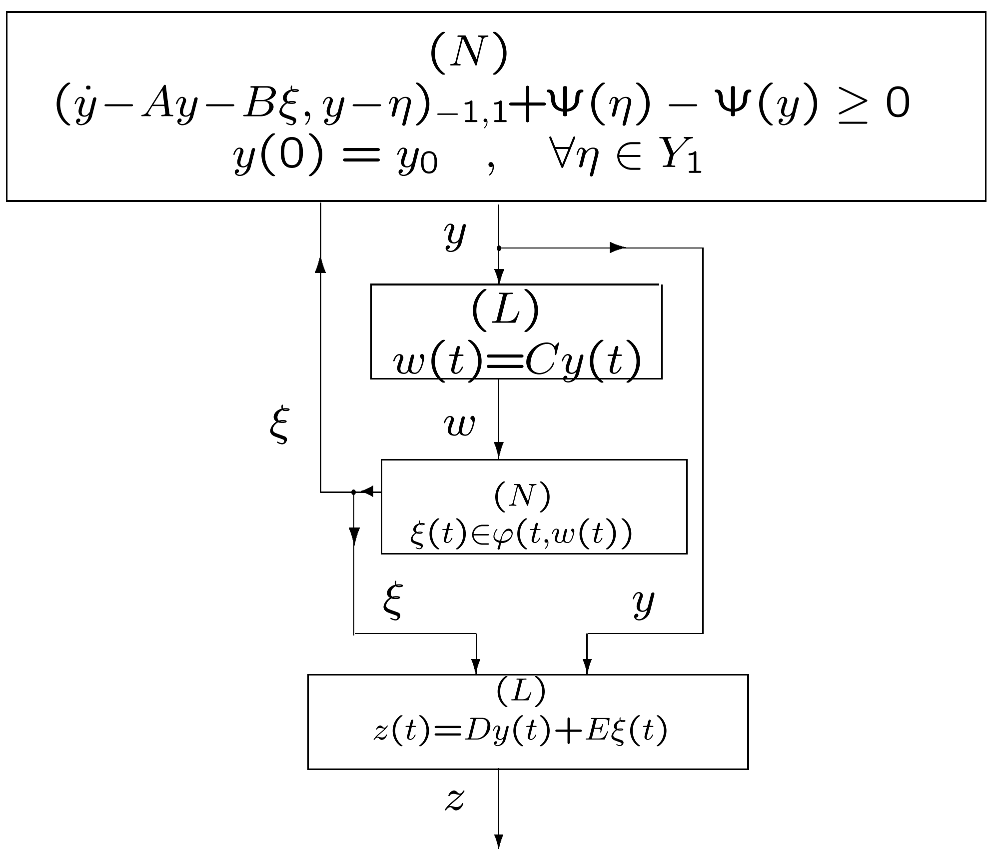

Fig. 1 State / linear output / nonlinear output / observation diagram

Definition 1 Any pair of functions $\{y(\cdot), \xi(\cdot)\}$ with $y \in \mathcal{L}_T$ and $\xi \in L^2_{\mathrm{loc}}(0,\infty;\Xi)$ such that $B\xi \in \mathcal{L}_T$, satisfying (6), (7) almost everywhere on $(0,T)$, is called **solution of the Cauchy problem** $y(0) = y_0,\ \xi(0) = \xi_0$ defined for (6), (7) .

Assumptions:

(C1) The Cauchy-problem (6), (7) has for arbitrary $y_0 \in Y_0$ and $\xi_0 \in \mathcal{E}(y_0) \subset \Xi$ at least one solution $\{y(\cdot), \xi(\cdot)\}$.

(C2) a) The nonlinearity $\varphi : \mathbb{R}_+ \times W \to \Xi$ is a function having the property that $\mathcal{A}(t) := -A - B\varphi(t, C\cdot) : Y_1 \to Y_{-1}$ is a family of monotone hemicontinuous operators such that the inequality

$$\|\mathcal{A}(t)y\|_{-1} \le c_1 \|y\|_1 + c_2\,, \quad \forall\, y \in Y_1\,,$$

is satisfied, where $c_1 > 0$ and $c_2 \in \mathbb{R}$ are constants not depending on $t \in [0,T]$.
For any $y \in Y_1$ and for any bounded set $U \subset Y_1$ the family of functions $\{(\mathcal{A}(t)\eta, y)_{-1,1}\,,\ \eta \in U\}$ is equicontinuous with respect to t on any compact subinterval of $\mathbb{R}_+$.
b) ψ is a proper, convex, and semicontinuous from below function on $\mathcal{D}(\psi) \subset Y_1$.

(C3) $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+; Y_{-1})$.

(C4) Consider only solutions y of (6),(7) for which $\dot{y}$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}; Y_{-1})$.

Remark 1 When $\psi \equiv 0$ in (6) the evolutionary variational inequality is equivalent for a.a. $t \in [0,T]$ to the equation

$$\begin{aligned}
&\dot{y} = Ay + B\xi + f(t) \quad \text{in } Y_{-1}\,,\\
&y(0) = Y_0\,,\ w(t) = Cy(t)\,,\ \xi(t) \in \varphi(t, w(t))\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \xi(0) \in \mathcal{E}(y_0)\,,\\
&z(t) = Dy(t) + E\xi(t).
\end{aligned}$$

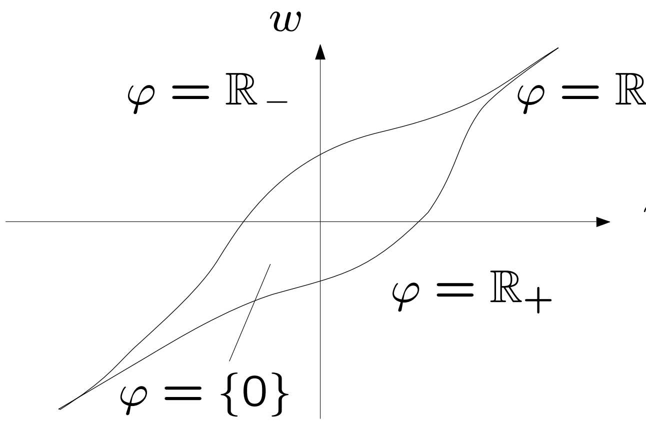

Fig. 2
Generalized play operator

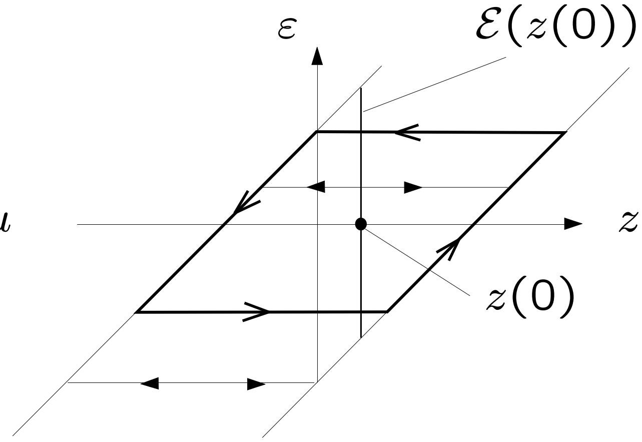

Fig. 3
Play (model of plasticity with strain-hardening)

Definition 2 a) Suppose F and G are quadratic forms on $Y_1 \times \Xi$. The **class of nonlinearities** $\mathcal{N}(F,G)$ defined by F and G consists of all maps $\varphi : \mathbb{R}_+ \times W \to 2^{\Xi}$ such that for any $y(\cdot) \in L^2_{\text{loc}}(0,\infty;Y_1)$ with $\dot{y}(\cdot) \in L^2_{\text{loc}}(0,\infty;Y_{-1})$ and any $\xi(\cdot) \in L^2_{\text{loc}}(0,\infty;\Xi)$ with $\xi(t) \in \varphi(t, Cy(t))$ for a.e. $t \geq 0$, it follows that $F(y(t),\xi(t)) \geq 0$ for a.e. $t \geq 0$ and (for any such pair $\{y,\xi\}$) there exists a continuous functional $\Phi : W \to \mathbb{R}$ such that for any times $0 \leq s < t$ we have

$$\int_s^t G(y(\tau),\xi(\tau))d\tau \geq \Phi(Cy(t)) - \Phi(Cy(s)) \, .$$

b) The **class of functionals** $\mathcal{M}(d)$ defined by a constant $d > 0$ consists of all maps $\psi : Y_1 \to \mathbb{R}_+$ such that for any $y \in L^2_{\text{loc}}(0,\infty;Y_0)$ with $\dot{y} \in L^2_{\text{loc}}(0,\infty;Y_1)$ the function

$t \mapsto \psi(y(t))$ belongs to $L^1(0,\infty;\mathbb{R})$ satisfying $\int_0^\infty \psi(y(t))dt \leq d$

and for any $\varphi \in \mathcal{N}(F,G)$ and any $\psi \in \mathcal{M}(d)$ the Cauchy-problem (6) $-$ (8) has a solution $\{y(\cdot),\xi(\cdot)\}$ on any time interval $[0,T]$.

3. Further assumptions

(F1) $A \in \mathcal{L}(Y_1, Y_{-1})$ is *regular*, i.e., for any $T > 0, y_0 \in Y_1$, $\psi_T \in Y_1$ and $f \in L^2(0, T; Y_0)$ the solutions of the direct problem

$$\dot{y} = Ay + f(t)\,,\; y(0) = y_0\,, \quad \text{a.a. } t \in [0, T]$$

and of the dual problem

$$\dot{\psi} = -A^*\psi + f(t)\,,\; \psi(T) = \psi_T\,, \quad \text{a.a. } t \in [0, T]$$

are strongly continuous in t in the norm of Y_1.
$A^* \in \mathcal{L}(Y_{-1}, Y_0)$ denotes the adjoint to A, i.e.,
$(Ay, \eta)_{-1,1} = (y, A^*\eta)_{-1,1}\,,\ \forall\, y, \eta \in Y_1\,.$

(F2) The pair (A, B) is L^2-*controllable*, i.e., for arbitrary $y_0 \in Y_0$ exists a control $\xi(\cdot) \in L^2(0, \infty; \Xi)$ such that the problem

$$\dot{y} = Ay + B\xi\,, \quad y(0) = y_0$$

is well-posed on the semiaxis $[0, +\infty)$, i.e., there exists a solution $y(\cdot) \in \mathcal{L}_\infty$ with $y\,(0) = y_0\,.$

(F3) $F(y, \xi)$ is an Hermitian form on $Y_1 \times \Xi$, i.e.,

$$F(y, \xi) = (F_1 y, y)_{-1,1} + 2\,\mathrm{Re}\,(F_2 y, \xi)_\Xi + (F_3\,\xi, \xi)_\Xi,$$

where

$$F_1 = F_1^* \in \mathcal{L}(Y_1, Y_{-1})\,,\; F_2 \in \mathcal{L}(Y_0, \Xi)\,,\; F_3 = F_3^* \in \mathcal{L}(\Xi, \Xi)\,.$$

Define the *frequency-domain condition* [Likhtarnikov and Yakubovich, 1976]

$$\alpha := \sup_{\omega, y, \xi} (\|y\|_1^2 + \|\xi\|_\Xi^2)^{-1} F(y, \xi)\,,$$

where the supremum is taken over all triples $(\omega, y, \xi) \in \mathbb{R}_+ \times Y_1 \times \Xi$ such that $i\omega y = Ay + B\xi\,.$

4. Absolute observation - stability of evolutionary inequalities

For a function $z(\cdot) \in L^2(\mathbb{R}_+; Z)$ we denote their norm by

$$\|z(\cdot)\|_{2,Z} := \left(\int_0^\infty \|z(t)\|_Z^2 \, dt \right)^{1/2} .$$

Definition 3 a) The inequality (6), (7) is said to be **absolutely dichotomic** (i.e., in the classes $\mathcal{N}(F,G), \mathcal{M}(d)$) **with respect to the observation** z from (8) if for any solution $\{y(\cdot), \xi(\cdot)\}$ of (6), (7) with $y(0) = y_0$, $\xi(0) = \xi_0 \in \mathcal{E}(y_0)$ the following is true: Either $y(\cdot)$ is unbounded on $[0, \infty)$ in the Y_0-norm or $y(\cdot)$ is bounded in Y_0 in this norm and there exist constants c_1 and c_2 (which depend only on $A, B, \mathcal{N}(F,G)$ and $\mathcal{M}(d)$) such that

$$\|Dy(\cdot) + E\xi(\cdot)\|_{2,Z}^2 \le c_1(\|y_0\|_0^2 + c_2) . \tag{9}$$

b) The inequality (6), (7) is said to be **absolutely stable with respect to the observation** z from (8) if (9) holds for any solution $\{y(\cdot), \xi(\cdot)\}$ of (6), (7).

Definition 4 The inequality (6)$-$(8) with $f \equiv 0$ is said to be **minimally stable** if the resulting equation for $\psi \equiv 0$ is minimally stable, i.e., there exists a bounded linear operator $K : Y_1 \to \Xi$ such that the operator $A + BK$ is stable, i.e. for some $\varepsilon > 0$

$$\sigma(A + BK) \subset \{s \in \mathbb{C} : \mathrm{Re}\, s \le -\varepsilon < 0\}$$

$$\text{with} \qquad F(y, Ky) \ge 0 , \qquad \forall\, y \in Y_1 , \tag{10}$$

$$\text{and} \qquad \int_s^t G(y(\tau), \; Ky(\tau)) d\tau \ge 0 ,$$

$$\forall\, s, t : 0 \le s < t , \quad \forall\, y \in L^2_{\mathrm{loc}}(\mathbb{R}_+; Y_1) . \tag{11}$$

Theorem 1 Consider the evolution problem (6) $-$ (8) with $\varphi \in \mathcal{N}(F,G)$ and $\psi \in \mathcal{M}(d)$. Suppose that for the operators A^c, B^c the assumptions **(F1)** and **(F2)** are satisfied. Suppose also that there exist an $\alpha > 0$ such that with the transfer operator

$$\chi^{(z)}(s) = D^c(sI^c - A^c)^{-1}B^c + E^c \qquad (s \notin \sigma(A^c)) \tag{12}$$

the frequency-domain condition

$$F^c\left((i\omega I^c - A^c)^{-1}B^c\xi, \xi\right)$$

$$+\, G^c\left((i\omega I^c - A^c)^{-1}B^c\xi, \xi\right) \le -\alpha \|\chi^{(z)}(i\omega)\xi\|^2_{Z^c}$$

$$\forall\, \omega \in \mathbb{R} : i\omega \notin \sigma(A^c)\,, \quad \forall\, \xi \in \Xi^c$$

is satisfied and the functional

$$J(y(\cdot),\xi(\cdot)) := \int_0^\infty [F^c(y(\tau),\xi(\tau)) + G^c(y(\tau),\xi(\tau))$$

$$+\, \alpha\|D^c y(\tau) + E^c\xi(\tau)\|^2_{Z^c}]\, d\tau$$

is bounded from above on any set

$$\mathbf{M}_{y_0} := \{y(\cdot),\xi(\cdot) : \dot{y} = Ay + B\xi \quad \text{on} \quad \mathbb{R}_+,$$

$$y(0) = \; y_0\,,\, y(\cdot) \in \mathcal{L}_\infty\,,\, \xi(\cdot) \in L^2(0,\infty;\Xi)\}\;.$$

Suppose further that the inequality (6)$-$(8) with $f \equiv 0$ is minimally stable, i.e., (10) and (11) are satisfied with some operator $K \in \mathcal{L}(Y_1, \Xi)$ and that the pair $(A + BK, D + EK)$ is observable in the sense of Kalman, i.e., for any solution $y(\cdot)$ of

$$\dot{y} = (A + BK)y\,, \quad y(0) = y_0\,,$$

with $z(t) = (D + EK)y\,(t) = 0$ for a.a. $t \ge 0$ it follows that $y(0) = y_0 = 0\,.$
Then inequality (6), (7) is absolutely stable with respect to the observation z from (8).

Proof: Reitmann, V. and H. Kantz, Observation stability of controlled evolutionary variational inequalities. Preprint-Series DFG-SPP 1114, Preprint 21, Bremen, 2003.

5. Application of observation stability to the beam equation

Consider the equation of a beam of length l, with damping and Hookean material, given as

$$\rho\,\mathbf{A}\frac{\partial^2 u}{\partial t^2}+\gamma\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\frac{\mathbf{EA}}{3}\tilde{g}\left(\frac{\partial u}{\partial x}\right)\right)=0\,, \tag{13}$$

$$u(0,t)=u(l,t)=0\quad\text{for}\quad t>0\,, \tag{14}$$

$$u(x,0)=u_0(x)\,,\;u_t(x,0)=u_1(x)\quad\text{for}\quad x\in(0,l)\,. \tag{15}$$

Here u is the deformation in the x direction. Assume that the cross section area $\mathbf{A}$, the viscose damping γ, the mass density ρ and the generalized modulus of elasticity $\mathbf{E}$ are constant. The nonlinear stress-strain law $\tilde{g}$, is given by

$$\tilde{g}(w)=1+w-(1+w)^{-2}\,,\quad w\in(-1,1)\,. \tag{16}$$

Assume that $\tilde{g}(w)=g(w)+w$.

$$\rho\mathbf{A}\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(\frac{\mathbf{EA}}{3}\frac{\partial u}{\partial x}\right)+\gamma\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\frac{\mathbf{EA}}{3}g\left(\frac{\partial u}{\partial x}\right)\right)=0\,. \tag{17}$$

Assume also that $\mathcal{V}_1\subset\mathcal{V}_0\subset\mathcal{V}_{-1}$ is a Gelfand triple with

$$\mathcal{V}_0:=L^2(0,l)\,,\;\mathcal{V}_1:=H_0^1(0,l)\text{ and }\mathcal{V}_{-1}:=H^{-1}(0,l)\,. \tag{18}$$

Then equation (13) $-$ (15) can be rewritten in $\mathcal{V}_{-1}$ as

$$\rho\mathbf{A}u_{tt}+\mathcal{A}_1u+\mathcal{A}_2u_t+\mathcal{C}^*g(\mathcal{C}u)=0\,, \tag{19}$$

$$u(0)=u_0\,,\quad u_t(0)=u_1\,, \tag{20}$$

with $\mathcal{A}_1\in\mathcal{L}(\mathcal{V}_1,\mathcal{V}_{-1})$, $\mathcal{A}_2\in\mathcal{L}(\mathcal{V}_1,\mathcal{V}_{-1})$ (strong damping), $\mathcal{C}\in\mathcal{L}(\mathcal{V}_1,\mathcal{V}_0)$ and $g:\mathcal{V}_0\to\mathcal{V}_0$. The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are associated with their bilinear forms $a_i:\mathcal{V}_1\times\mathcal{V}_1\to\mathbb{R}$ $(i=1,2)$ through $(\mathcal{A}_iv,w)_{\mathcal{V}_{-1},\mathcal{V}_1}=a_i(v,w)\,,\;\forall\,v,w\in\mathcal{V}_0$.

Assumptions:

(A1) a) The form a_1 is symmetric on $\mathcal{V}_0 \times \mathcal{V}_;$

b) a_1 is $\mathcal{V}_1$ continuous, i.e., for some $c_1 > 0$ holds $|a_1(v,w)| \leq c_1 \|v\|_{\mathcal{V}_1} \|w\|_{\mathcal{V}_1}, \quad \forall\, v, w \in \mathcal{V}_1$;

c) a_1 is strictly $\mathcal{V}_1$-elliptic, i.e., for some $k_1 > 0$ holds $a_1(v,v) \geq k_1 \|v\|^2_{\mathcal{V}_1}, \quad \forall\, v \in \mathcal{V}_1$.

(A2) a) The form a_2 is $\mathcal{V}_1$ continuous, i.e., for some $c_2 > 0$ holds $|a_2(v,w)| \leq c_2 \|v\|_{\mathcal{V}_1} \|w\|_{\mathcal{V}_1}$, $\forall\, v, w \in \mathcal{V}_1$.

b) The form a_2 is $\mathcal{V}_1$ coercive and symmetric, i.e., there are $k_2 > 0$ and $\lambda_0 \geq 0$ s.t.

$a_2(v,v) + \lambda_0 \|v\|^2_{\mathcal{V}_0} \geq k_2 \|v\|^2_{\mathcal{V}_1}$ and

$a_2(v,w) = a_2(w,v)\,, \quad \forall\, v, w \in \mathcal{V}_1$.

(A3) a) The operator $\mathcal{C} \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)$ satisfies with some $k \geq 0$ the inequality

$$\|\mathcal{C}v\|_{\mathcal{V}_0} \leq \sqrt{k} \|v\|_{\mathcal{V}_1}, \quad \forall\, v \in \mathcal{V}_1 .$$

$g : \mathcal{V}_0 \to \mathcal{V}_0$ is continuous and $\|g(v)\|_{\mathcal{V}_0} \leq c_1 \|v\|_{\mathcal{V}_0} + c_2$ for $v \in \mathcal{V}_0$, where c_1 and c_2 are nonnegative constants.

b) g is of gradient type, i.e., there exists a coninuous Frechét-differentiable functional $G : \mathcal{V}_0 \to \mathbb{R}$, whose Frechét derivative $G^{'}(v) \in \mathcal{L}(\mathcal{V}_0\,, \mathbb{R})$ at any $v \in \mathcal{V}_0$ can be represented in the form

$$G^{'}(v)w = (g(v), w)_{\mathcal{V}_0}, \quad \forall\, w \in \mathcal{V}_0 .$$

c) $g(0) = 0$ and for some positive $\varepsilon < 1$ we have for all $v, w \in \mathcal{V}_0$

$$(g(v) - g(w), v - w)_{\mathcal{V}_0} \geq -\varepsilon k_1 k^{-1} \|v - w\|^2_{\mathcal{V}_0} . \tag{21}$$

Definition 5 We say that $u \in \mathcal{L}_T$ is a *weak solution* of (19), (20) if

$$(u_{tt}, \eta)_{\mathcal{V}_{-1},\mathcal{V}_1} + a_1(u,\eta) + a_2(u_t,\eta) + (g(\mathcal{C}u), \mathcal{C}u)_0 = 0 \quad (22)$$

$$\forall\, \eta \in \mathcal{L}_T\,,\ \text{a.a.}\, t \in [0,T]\,.$$

Introduce $Y_0 := \mathcal{V}_1 \times \mathcal{V}_0$ in the coordinates $y = (y_1, y_2) = (u, u_t)$. Define for this $Y_1 := \mathcal{V}_1 \times \mathcal{V}_1$ and $a : Y_1 \times Y_1 \to \mathbb{R}$ by

$$a((v_1,v_2),(w_1,w_2)) = (v_2,w_1)_{\mathcal{V}_1} - a_1(v_1,w_2) - a_2(v_2,w_2)\,,$$

$$\forall\, (v_1,v_2),(w_1,w_2) \in Y_1 \times Y_1\,. \quad (23)$$

The norms in the product spaces Y_0 and Y_1 are

$$\|(y_1,y_2)\|_0^2 := \|y_1\|_{\mathcal{V}_1}^2 + \|y_2\|_{\mathcal{V}_0}^2\,, \quad (y_1,y_2) \in Y_0\,,$$

and

$$\|(y_1,y_2)\|_1^2 := \|y_1\|_{\mathcal{V}_1}^2 + \|y_2\|_{\mathcal{V}_1}^2\,, \quad (y_1,y_2) \in Y_1\,.$$

Then (22) can be rewritten as

$$(\dot y, \eta)_{-1,1} - a(y,\eta) = (B\varphi(Cy), \eta)_{-1,1}\,, y(0) = (u_0,u_1)\,,$$

$$\forall\, \eta \in Y_1\,, \quad (24)$$

where $$B\varphi(Cy) := \begin{pmatrix} 0 \\ -\mathcal{C}^* g\,(\mathcal{C}y_1) \end{pmatrix}\,, \quad (25)$$

$$\dot y = Ay + B\varphi\,(Cy),\ y(0) = y_0\,, \quad (26)$$

$$a(v,w) = (Av,w)_{-1,1}\,, \quad \forall\, v,w \in Y_1\,, \quad \text{i.e.,} A = \begin{bmatrix} 0 & I \\ -\mathcal{A}_1 & -\mathcal{A}_2 \end{bmatrix}.$$

$Y_1 \subset Y_0$ is completely continuous, A generates an analytic semigroup on Y_1, Y_0 and $Y_{-1} = \mathcal{V}_1 \times \mathcal{V}_{-1}$.

The semigroup is exponentially stable on Y_1, Y_0 and Y_{-1}, the pair (A, B) is exponentially stabilizable.
Consider with parameters $\varepsilon > 0$ and $\alpha \in \mathbb{R}$

$$\frac{\partial^2 u}{\partial t^2} + 2\varepsilon\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = -\alpha\left(\frac{\partial}{\partial x}\left(-g\left(\frac{\partial u}{\partial x}\right)\right)\right) =: \alpha\frac{\partial}{\partial x}\,\xi\,, \tag{27}$$

the boundary and initial conditions (14), (15), where $\xi = -g = \varphi$ is introduced as new nonlinearity.
Assume that $\varphi \in \mathcal{N}(F)$, with the quadratic form $F(w, \xi) = \mu w^2 - \xi w$ on $\mathbb{R} \times \mathbb{R}$, where $\mu > 0$ is a certain parameter.
$\lambda_k > 0$ and e_k $(k = 1, 2, \ldots)$ are the eigenvalues resp. eigenfunctions of the operator $-\Delta$ with zero boundary conditions.
Write formally the Fourier series of the solution $u(x, t)$ and the perturbation $\xi\,(x, t)$ to the (linear) equation (27) as

$$u\,(x, t) = \sum_{k=1}^{\infty} u^k(t)e_k \quad \text{and} \quad \xi\,(x, t) = \sum_{k=1}^{\infty} \xi^k(t)e_k\,. \tag{28}$$

Introduce the Fourier transforms $\tilde{u}$ and $\tilde{\xi}$ of (28) with respect to the time variable. From (27) for $k = 1, 2, \ldots$ it follows that

$$-\omega^2\tilde{u}^k\,(i\omega) + 2i\omega\varepsilon\tilde{u}^k\,(i\omega) + \lambda_k\tilde{u}^k(i\omega) = -\alpha\sqrt{\lambda_k}\,\tilde{\xi}^k\,(i\omega)\,, \tag{29}$$

$$\tilde{u}^k = \chi\,(i\omega, \lambda_k)\,\tilde{\xi}^k\,, \tag{30}$$

$$\begin{aligned}\chi\,(i\omega, \lambda_k) = (-\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k)^{-1}\,(\alpha\lambda_k)\,,\\ \forall\,\omega \in \mathbb{R} : -\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k \neq 0\,.\end{aligned} \tag{31}$$

Consider the functional

$$J(w, \xi) := \text{Re}\int_0^{\infty}\int_0^l (\mu|w|^2 - w\xi^*)\,dxdt\,. \tag{32}$$

The Parseval equality for (32) gives

$$|\tilde{w}|^2 = \sum_{k=1}^{\infty} \lambda_k |\tilde{u}^k|^2 = \sum_{k=1}^{\infty} \lambda_k |\tilde{u}^k|^2 = \sum_{k=1}^{\infty} \lambda_k |\chi(i\omega, \lambda_k)|^2 |\tilde{\xi}^k|^2$$

$$\text{and} \quad \tilde{w}\,\tilde{\xi}^* = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\tilde{u}^k\,(\tilde{\xi}^k)^* = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\chi\,(i\omega, \lambda_k) |\tilde{\xi}^k|^2 \,.$$

Then [Arov and Yakubovich, 1982] the functional (32) is bounded from above if and only if the functional

$$\text{Re} \int_{-\infty}^{+\infty} \int_0^l [\mu \left(\sum_{k=1}^{\infty} \lambda_k |\chi(i\omega, \lambda_k)|^2 |\tilde{\xi}^k|^2 - \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\chi\,(i\omega, \lambda_k) |\tilde{\xi}^k|^2 \right)] dx d\omega \tag{33}$$

is bounded on the subspace of Fourier-transforms from (30), (31) or the frequency-domain condition

$$\mu\lambda_k |\chi\,(i\omega, \lambda_k)|^2 - \sqrt{\lambda_k}\,\text{Re}\,\chi(i\omega, \lambda_k) < 0\,, \tag{34}$$
$$\forall\,\omega \in \mathbb{R} : -\omega^2 + 2\,i\omega\varepsilon + \alpha\lambda_k \neq 0\,,\ k = 1, 2, \ldots,$$

is satisfied, where $\chi\,(i\omega, \lambda_k) = (-\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k)^{-1}(-\alpha\sqrt{\lambda_k})\,.$