On a generalization of Leonov's invariant cones method for boundary control problems

V. Reitmann *

Saint-Petersburg State University, Russia
Department of Mathematics and Mechanics

Equadiff 07, August 5 - 11, 2007, Vienna

*Supported by DAAD (German Academic Exchange Service)

1. Control systems in Lur'e form with a Duffing type nonlinearity

Let $\mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1}$ be a Gelfand rigging of the real Hilbert space $\mathcal{V}_0$, i.e. a chain of Hilbert spaces with dense and continuous inclusions. Denote by $(\cdot,\cdot)_{\mathcal{V}_j}$ and $\|\cdot\|_{\mathcal{V}_j}, j = 1, 0, -1$, the scalar product resp. norm in $\mathcal{V}_j (j = 1, 0, -1)$ and by $(\cdot,\cdot)_{\mathcal{V}_{-1},\mathcal{V}_1}$ the pairing between $\mathcal{V}_{-1}$ and $\mathcal{V}_1$.

Let $A_0 \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_{-1})$ be a linear operator, $b_0 \in \mathcal{V}_{-1}$ a generalized vector, $c_0 \in \mathcal{V}_0$ a vector and $d_0 < 0$ a number. According to the vectors c_0 and b_0 we introduce the linear operators $C_0 \in \mathcal{L}(\mathcal{V}_0, \mathbb{R})$ and $B_0 \in \mathcal{L}(\mathbb{R}, \mathcal{V}_{-1})$ by $C_0\nu = (c_0, \nu)_{\mathcal{V}_0}, \ \forall\, \nu \in \mathcal{V}_0$, and $B_0\xi := \xi b_0, \ \forall \xi \in \mathbb{R}$.

Assume that $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two scalar-valued functions. Our aim is to study a system of indirect control, which is formally given as

$$\begin{aligned} \dot{\nu} &= A_0\nu + b_0[\phi(t,z) + g(t)]\,, \\ \dot{z} &= (c_0,\nu)_{\mathcal{V}_0} + d_0[\phi(t,z) + g(t)]\,. \end{aligned} \tag{1}$$

Let us demonstrate how (1) can be written as a standard control system. Consider for this the Gelfand rigging $Y_1 \subset Y_0 \subset Y_{-1}$, in which

$$Y_j := \mathcal{V}_j \times \mathbb{R}\,, \quad j = 1, 0, -1\,. \tag{2}$$

The scalar product $(\cdot,\cdot)_j$ in Y_j is introduced as

$\big((\nu_1, z_1), (\nu_2, z_2)\big)_j := (\nu_1, \nu_2)_{\mathcal{V}_j} + z_1 z_2$, where $(\nu_1, z_1), (\nu_2, z_2) \in Y_j$ are arbitrary. The pairing between Y_{-1} and Y_1 is defined for $(h, \xi) \in \mathcal{V}_{-1} \times \mathbb{R} = Y_{-1}$ and $(\nu, \varsigma) \in \mathcal{V}_1 \times \mathbb{R} = Y_1$ through

$$((h,\xi),(\nu,\varsigma))_{-1,1} := (h,\nu)_{\mathcal{V}_{-1},\mathcal{V}_1} + \xi\,\varsigma\,. \tag{3}$$

Let $b := \begin{bmatrix} b_0 \\ d_0 \end{bmatrix} \in Y_{-1}$ and $c := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in Y_0$. Suppose further that the operators $C \in \mathcal{L}(Y_0, \mathbb{R})$ and $B \in \mathcal{L}(\mathbb{R}, Y_{-1})$ are given as

$$Cy = (c, y)_0\,, \quad \forall\, y \in Y_0\,, \qquad B\xi = \xi b\,, \quad \forall\, \xi \in \mathbb{R}\,,$$

and the operator $A \in \mathcal{L}(Y_1, Y_{-1})$ is defined as

$$A := \begin{bmatrix} A_0 & 0 \\ C_0 & 0 \end{bmatrix} .$$

Consider now the system

$$\dot{y} = Ay + B\left[\phi(t, z) + g(t)\right], \; z = Cy \; , \tag{4}$$

which is equivalent to (1) through $y = (\nu, z)$.

If $-\infty \leq T_1 < T_2 \leq +\infty$ are arbitrary, we define the norm for Bochner measurable functions in $L^2(T_1, T_2; Y_j)$, $j = 1, 0, -1$, by

$$\|y\|_{2,j} := \left(\int_{T_1}^{T_2} \|y(t)\|_j^2 \, dt \right)^{1/2} . \tag{5}$$

Let $\mathcal{W}(T_1, T_2; Y_1, Y_{-1})$ be the space of functions y such that

$y \in L^2(T_1, T_2; Y_1)$ and $\dot{y} \in L^2(T_1, T_2; Y_{-1})$, equipped with the norm

$$\|y\|_{\mathcal{W}(T_1,T_2;Y_1,Y_{-1})} := \left(\|y\|_{2,1}^2 + \|\dot{y}\|_{2,-1}^2 \right)^{1/2} . \tag{6}$$

Let us introduce the following assumptions **(A1)** – **(A6)** about the operator $A_0 \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_{-1})$, the vectors $b_0 \in \mathcal{V}_{-1}$ and $c_0 \in \mathcal{V}_0$, and the functions ϕ and g.

(A1) For any $T > 0$ and any

$(f_1, f_2) \in L^2(0, T; \mathcal{V}_{-1} \times \mathbb{R})$ the problem

$$\begin{aligned} &\dot{\nu} = A_0 \nu + f_1(t) \, , \\ &\dot{z} = (c_0, \nu)_{\mathcal{V}_0} + f_2(t) \; , \quad (\nu(0), z(0)) = (\nu_0, z_0) \end{aligned} \tag{7}$$

is well-posed, i.e. for arbitrary

$(\nu_0, z_0) \in Y_0$, $(f_1, f_2) \in L^2(0,T;\mathcal{V}_{-1} \times \mathbb{R})$ there exists a unique solution $(\nu, z) \in \mathcal{W}(0,T;Y_1,Y_{-1})$ satisfying (7) in a variational sense and depending continuously on the initial data, i.e.

$$\|(\nu, z)\|^2_{\mathcal{W}(0,T;Y_1,Y_{-1})} \leq k_1\|(\nu_0, z_0)\|^2_{\mathcal{V}_0\times\mathbb{R}} + k_2\|(f_1, f_2)\|^2_{2,-1}\,, \tag{8}$$

where $k_1 > 0$ and $k_2 > 0$ are some constants.

(A2) There is a $\lambda > 0$ such that $A_0 + \lambda I$ is a Hurwitz operator.

(A3) For any $T > 0, (\nu_0, z_0) \in \mathcal{V}_1 \times \mathbb{R}, (\tilde{\nu}_0, \tilde{z}_0) \in \mathcal{V}_1 \times \mathbb{R}$ and $(f_1, f_2) \in L^2(0,T;\mathcal{V}_1 \times \mathbb{R})$ the solution of the direct problem (7) and the solution of the adjoint problem

$$\begin{aligned}\dot{\tilde{\nu}} &= -(A_0^+ + \lambda\, I)\tilde{\nu} + f_1(t)\\ \dot{\tilde{z}} &= -C_0^+\tilde{z} - \lambda\,\tilde{z} + f_2(t)\end{aligned} \tag{9}$$

are strongly continuous in t in the norm of $\mathcal{V}_1 \times \mathbb{R}$.

(A4) The pair (A_0, b_0) is L^2-controllable, i.e. for arbitrary $\nu_0 \in \mathcal{V}_0$ there exists a control $\xi\,(\cdot) \in L^2(0,\infty;\mathbb{R})$ such that the problem

$$\dot{\nu} = A_0\nu + b_0\xi\,, \quad \nu(0) = \nu_0$$

is well-posed in the variational sense on $(0,\infty)$.

Introduce by (c denotes the complexification)

$$\chi(p) = \left(c_0^c, (A_0^c - pI^c)^{-1}\, b_0^c\right)_{\mathcal{V}_0}\,, \quad p \in \rho(A_0^c)$$

the transfer function of the triplet (A_0^c, b_0^c, c_0^c).

(A5) Suppose $\lambda > 0$ and $\kappa_1 > 0$ are parameters, where λ is from **(A2)**. Then:

$$\begin{aligned}a)\ &\lambda d_0 + \mathrm{Re}\,(-i\omega - \lambda)\chi(i\omega - \lambda) +\\ &\kappa_1\,|\,\chi(i\omega - \lambda) - d_0\,|^2 \leq 0\,, \qquad \forall\,\omega \geq 0\,.\end{aligned} \tag{10}$$

(A6) The function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\phi(t, 0) = 0$, $\forall\, t \in \mathbb{R}$. The function $g : \mathbb{R} \to \mathbb{R}$ belongs to $L^2_{\text{loc}}(\mathbb{R}\,;\mathbb{R})$. There are numbers $\kappa_1 > 0$ (from **(A5)**), $0 \le \kappa_2 < \kappa_3 < +\infty, \beta_1 < \beta_2$ and $\zeta_2 < \zeta_1$ such that:

$$a) \quad \beta_1 < g(t) < \beta_2\,, \tag{11}$$

for a.a. t from an arbitrary compact time interval ;

$$b) \quad (\phi(t,z) + \beta_i)(z - \zeta_i) \le \kappa_1 (z - \zeta_i)^2, \quad i = 1, 2\,,$$

$$\forall\, t \in \mathbb{R}, \quad \forall\, z \in [\zeta_2, \zeta_1]\,; \tag{11a}$$

$$c) \quad \kappa_2 (z_1 - z_2)^2 \le (\phi(t, z_1) - \phi(t, z_2))(z_1 - z_2) \le$$

$$\kappa_3 (z_1 - z_2)^2\,, \qquad \forall\, t \in \mathbb{R},\ \forall\, z_1, z_2 \in [\zeta_2, \zeta_1]\,. \tag{11b}$$

Theorem 1 *Assume that for system (1) the hypotheses* **(A1)** – **(A6)** *are satisfied. Then there exists a closed, positively invariant and convex set* $\mathcal{G}$ *such that*

$$\{(\nu, z) \in \mathcal{V}_1 \times \mathbb{R} \,|\, \nu = 0, z \in [\zeta_2, \zeta_1]\} \subset \mathcal{G} \subset$$
$$\{(\nu, z) \in \mathcal{V}_1 \times \mathbb{R} \,|\, z \in [\zeta_2, \zeta_1]\}\ . \tag{12}$$

Suppose that $Y_1 \subset Y_0 \subset Y_{-1}$ is a Gelfand rigging of Y_0, $\|\cdot\|_j, (\cdot,\cdot)_j$ are the corresponding norms and scalar products, respectively, and $(\cdot,\cdot)_{-1,1}$ is the pairing between Y_{-1} and Y_1. Consider the linear system

$$\dot{y} = Ay\,, \quad z = (c, y)_0\,, \tag{13}$$

where $A \in \mathcal{L}(Y_1, Y_{-1})$ and $c \in Y_0$.

Assume that for each $y_0 \in Y_0$ there exists a unique solution $y(\cdot, y_0)$ of (13) in $\mathcal{W}(0, \infty)$ satisfying $y(0, y_0) = y_0$. In the sequel we need the following assumption.

Brusin, V. A. (1976). The Lur'e equations in Hilbert space and its solvability. *Prikl. Math. Mekh.* **40** (5), 947 – 955. (in Russian)

Likhtarnikov, A. L. and V. A. Yakubovich (1976). The frequency theorem for equations of evolutionary type. *Siberian Math. J.* **17**, 790 – 803.

(A7) The space Y_0 can be decomposed as $Y_0 = Y_0^+ \oplus Y_0^-$ such that the following holds:

a) For each $y_0 \in Y_0^+$ we have $\lim_{t\to\infty} y(t, y_0) = 0$.
For each $y_0 \in Y_0^-$ there exists a unique solution $y_-(t) = y(t, y_0)$ of (13), defined on $(-\infty, 0)$, such that $\lim_{t\to-\infty} y_-(t) = 0$ and $(c, y(t, y_0))_0 = 0, \forall t \geq 0$, if and only if $y_0 = 0$.

b) For each $y_0 \in Y_0^+$ the equality $(c, y(t, y_0))_0 = 0,\ \forall t \leq 0$, holds if and only if $y_0 = 0$.
For each $y_0 \in Y_0^-$ the equality $(c, y(t, y_0))_0 = 0,\ \forall t \leq 0$, holds if and only if $y_0 = 0$.

Lemma 1 *Suppose that system (13) satisfies* **(A7)** *and there exists a linear continuous operator* $P : Y_0 \to Y_0, P^* = P$, *such that for any* $s \leq t$ *and any solution* $y(\cdot, y_0)$ *of (13) we have with* $V(y) := (y, Py)_0, y \in Y_0$,

$$V(y(t, y_0)) - V(y(s, y_0)) \leq -\int_s^t (c, y(\tau, y_0))_0^2 \ d\tau \,. \tag{14}$$

Then

$$P_{|Y_0^+} \geq 0 \ , \ \textit{i.e.}, \ (y, Py)_0 > 0 \quad \textit{for all} \quad y \in Y_0^+ \backslash \{0\} \tag{15}$$

and

$$P_{|Y_0^-} \leq 0 \ , \ \textit{i.e.}, \ (y, Py)_0 < 0 \quad \textit{for all} \quad y \in Y_0^- \backslash \{0\} \,. \tag{16}$$

Assume that Y is a Hilbert space with scalar product $(\cdot,\cdot)$. A *cone* in Y is a set $\mathcal{C} \subset Y, \mathcal{C} \neq \varnothing$, such that $y \in \mathcal{C}, \alpha \in \mathbb{R}_+$ imply that $\alpha y \in \mathcal{C}$. It is easy to see that a cone $\mathcal{C}$ in Y is convex if and only if $y_1, y_2 \in \mathcal{C}$ imply that $y_1 + y_2 \in \mathcal{C}$.

Suppose that $P \in \mathcal{L}(Y), P = P^*$. Then the set

$$\mathcal{C} := \{y \in Y \,|\, (y, Py) \leq 0\}$$

is a cone which is called by us *quadratic.*

Assume that there is a decomposition $Y = Y^+ \oplus Y^-$ such that $P_{|Y^+} \geq 0$ and $P_{|Y^-} \leq 0$. Then the quadratic cone

$$\{y \in Y \,|\, (y, Py) \leq 0\}$$

is called by us *quadratic cone of dimension* $\dim Y^-$.

Lemma 2 *Suppose that:*

1) $Y_1 \subset Y_0 \subset Y_{-1}$ *is a Gelfand rigging of the Hilbert space* Y_0 *with scalar products* $(\cdot,\cdot)_i$, *corresponding norms* $\|\cdot\|_i, i = 1, 0, -1$, *and pairing* $(\cdot,\cdot)_{-1}$, *between* Y_{-1} *and* Y_1*;*

2) *There is an operator* $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$, *self-adjoint in* Y_0 *such that*

$$\mathcal{C} := \{y \in Y_0 \,|\, (y, Py)_0 \leq 0\}$$

is an 1*-dimensional quadratic cone;*

3) *There are vectors* $h \in Y_{-1}$ *and* $r \in Y_0$ *such that*

$$2\,(h, Py)_{-1,1} = (r, y)_0\,,\ \forall\, y \in Y_1 \tag{17}$$

$$\textit{and} \qquad (h, r)_{-1,1} < 0\,. \tag{18}$$

Then we have

$$\mathrm{int}\,\mathcal{C} \cap \{y \in Y_1 \,|\, (y, r)_0 = 0\} = \varnothing\,. \tag{19}$$

Blyagoz, Z.U. and G.A. Leonov (1978). Frequency criteria for stability in the large of nonlinear systems. *Vestn. Leningr. Univers.* **13**, 18 – 23. (in Russian)

Leonov, G.A. and A.N. Churilov (1976). Frequency-domain conditions for boundedness of solutions of phase systems. *Dynamics of systems, Meshvuz. Sb., Gorky* **10**, 3 – 20. (in Russian)

V. Reitmann (1982). Über die Beschränktheit der Lösungen nichtstationärer Phasensysteme. *ZAA* **1**, 83 – 93.

Really, in the finite-dimensional case we have $Y_1 = Y_0 = Y_{-1} = \mathbb{R}^n, (\cdot,\cdot)_{-1,1} = (\cdot,\cdot)_0 = (\cdot,\cdot)$ the Euclidean inner product and $P = P^*,\ \det P \neq 0$, a regular symmetric $n \times n$ matrix. Assumption (17) in Lemma 2 states that there are vectors $h, r \in \mathbb{R}^n$ such that

$$2\,(h, Py) = (r, y)\;, \quad \forall\, y \in \mathbb{R}^n\;. \tag{20}$$

It follows from (20) that

$$2\,h = P^{-1}r\;. \tag{21}$$

Equation (21) shows that assumption (18) of Lemma 2 takes the form

$$(r, P^{-1}r) < 0\;. \tag{22}$$

If (22) is satisfied, it follows from Lemma 2 for the 1-dimensional quadratic cone
$\mathcal{C} = \{y \in \mathbb{R}^n | (y, Py) \le 0\}$ that

$$\mathrm{int}\,\mathcal{C} \cap \{y \in \mathbb{R}^n \,|\, (y, r) = 0\} = \varnothing\;. \tag{23}$$

(A8) The imbedding $\mathcal{V}_1 \subset \mathcal{V}_0$ is compact.

(A9) The family of operators $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}, \mathcal{A}(t) : Y_1 \to Y_{-1}$, given by $\mathcal{A}(t)\eta := -A\eta - B\phi(t, C\eta), \forall t \in \mathbb{R}, \forall \eta \in Y_1$, is monotone on the segment $\{\eta \in Y_1 \,|\, C\eta \in [\zeta_2, \zeta_1]\}$, i.e. for any $t \in \mathbb{R}$ we have

$$(\mathcal{A}(t)\eta - \mathcal{A}(t)\vartheta, \eta - \vartheta)_{-1,1} \geq 0\,, \quad \forall\, \eta, \vartheta \in Y_1\,, \quad \text{such that} \quad C\eta, C\vartheta \in [\zeta_2, \zeta_1]\,. \tag{24}$$

Theorem 2 *Assume that for system (1) the assumptions* **(A1)** – **(A9)** *are satisfied. Then it holds:*

a) For any $g \in BS^2(\mathbb{R};\mathbb{R})$ and any $(\nu_0, z_0) \in \mathcal{G}$, where $\mathcal{G}$ is the associated positively invariant set, there exists a solution $(\nu, z) \in \mathcal{W}(0, \infty; \mathcal{V}_1 \times \mathbb{R}, \mathcal{V}_{-1} \times \mathbb{R})$ of (1) such that $(\nu(0), z(0)) = (\nu_0, z_0)$.

b) For any $g \in BS^2(\mathbb{R};\mathbb{R})$ there exists for (1) a solution

$$(\nu_*, z_*) \in C_b(\mathbb{R}\,; \mathcal{V}_0 \times \mathbb{R}) \cap BS^2(\mathbb{R}; \mathcal{V}_1 \times \mathbb{R})\,. \tag{25}$$

(A10) Any continuous function ϕ which satisfies (11a) and (11b) has a continuous extension to a function $\tilde{\tilde{\phi}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies (11a) and (11b) for all $(t, z) \in \mathbb{R} \times \mathbb{R}$.

Theorem 3 *Assume that for system (1) the assumptions* **(A1**) – **(A10)** *are satisfied and in addition to this the following holds:*

(i) The operator $\begin{bmatrix} A_0 & \kappa_2 B_0 \\ C_0 & \kappa_2 d_0 \end{bmatrix}$ from $\mathcal{L}(Y_1, Y_{-1})$ is Hurwitz;

(ii) There exists a number $\epsilon > 0$ such that

$$\frac{1}{\kappa_3 - \kappa_2} + \mathrm{Re}\,\frac{\chi(i\omega) - d_0}{i\omega + \kappa_2(\chi(i\omega) - d_0)} > \epsilon,\ \forall\, \omega \in \mathbb{R}\,. \tag{26}$$

Then we have:

a) For any $g \in BS^2(\mathbb{R};\mathbb{R})$ system (1) has a unique solution (ν_*, w_*) inside $\mathcal{G}$ which satisfies (25) and this solution is exponentially stable inside $\mathcal{G}$.

b) Let the families of functions $\{\phi(\cdot,z)\,|\,z \in [\zeta_2,\zeta_1]\}$ and $\{\tilde{\phi}(\cdot,z)\,|\,z \in \mathcal{S}\}$, where $\tilde{\phi}$ is from **(A9)** *and $\mathcal{S} \subset \mathbb{R}$ is an arbitrary bounded interval, be uniformly Bohr a.p. . Then for any S^2-a.p. forcing function g the unique in $\mathcal{G}$ bounded and exponentially stable solution (ν_*, z_*) is Bohr a.p. .*

2. Example

We consider the restricted *boundary control problem* for the temperature (Butkovskii, 1975)

$$\theta_t = \delta_1\theta_{xx} - \delta_2\,\theta\ ,\quad \delta_1 > 0, \delta_2 > 0 \tag{27}$$

$$\theta_{x_{|x=0}} = 0\,,\ \theta_{x_{|x=1}} = \delta_3[\phi(t,w) + g(t)]\,,\ \delta_3 \in \mathbb{R}\,,$$

$$\dot{w} = \int_0^1 \theta(x,t)\,k(x)\,dx + \delta_4[\phi(t,w) + g(t)]\ . \tag{28}$$

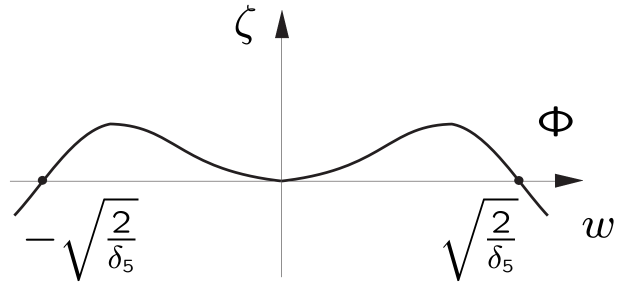
Here $k(\cdot)$ is a kernel function, $\quad \delta_4 < 0\,,$

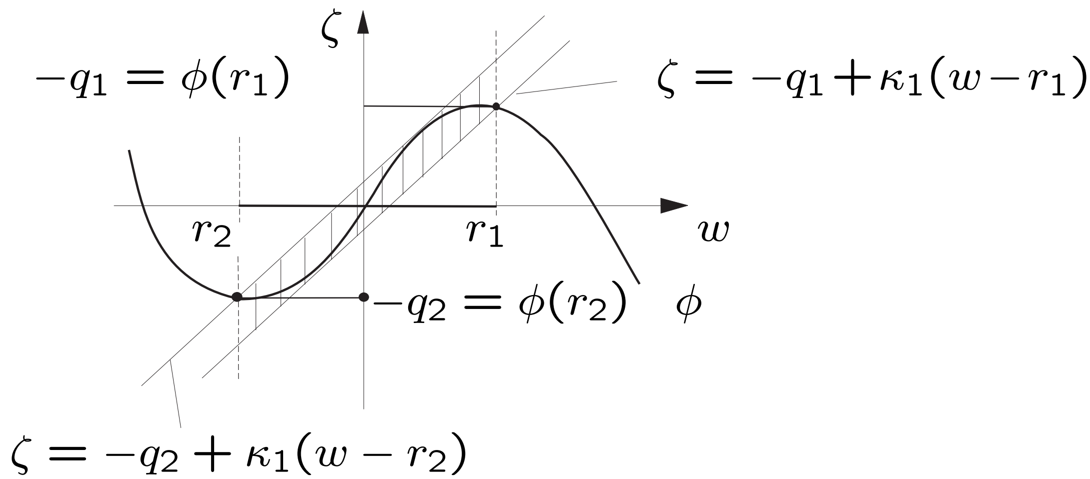
$\phi(t,w) = w - \delta_5(t)w^3$ is a *Duffing-type* nonlinearity, $\quad \delta_5(t) \geq 0$ a.e. (29)

Nonlinearity and forcing function:

$$\phi(w) = w - \delta_5 w^3\,,\ \delta_5 > 0$$

$$\phi = \Phi'\,,\qquad \Phi(w) = \frac{w^2}{2} - \frac{\delta_5}{4}w^4 \qquad\qquad \textit{double-well potential}$$





$$r_2 = -\frac{1}{\sqrt{3\,\delta_5}} + \varepsilon$$

$$r_1 = \frac{1}{\sqrt{3\,\delta_5}} - \varepsilon\,, \qquad \varepsilon > 0 \qquad \text{small}$$

$$q_2 = -\phi(r_2)\ ,\ q_1 = -\phi(r_1)$$

$$\kappa_1 = \frac{\delta_2^2}{4}$$

Write (27), (28) as ODE in Hilbert space

$$\dot{\nu} = A_0\nu + B_0[\phi(t,w) + g(t)] \tag{30}$$

$$\dot{w} = C_0\nu + d_0[\phi(t,w) + g(t)]\ , \tag{31}$$

$\mathcal{V}_1 := W^{1,2}(0,1)$, $\mathcal{V}_0 := L_2(0,1)$, $\mathcal{V}_{-1} = \mathcal{V}_1^*$,
space of test functions — state space (w.r.t. $\mathcal{V}_0$) — dual space

$$(\nu, \vartheta)_1 := \int_0^1 [\nu\vartheta + \nu_x\vartheta_x]\, dx\,, \qquad \nu, \vartheta \in \mathcal{V}_1\,.$$

$A_0 : \mathcal{V}_1 \to \mathcal{V}_{-1}$ is given by

$$(A_0\nu, \vartheta) = -\int_0^1 [\delta_1\nu'(x)\vartheta'(x) + \delta_2\nu(x)\vartheta(x)]\, dx\,.$$

$B_0 : \mathbb{R} \to \mathcal{V}_{-1}$ *(Control operator)* is given through

$$(B_0\,\xi, \nu) = \delta_1\xi\nu(1)\ , \quad \forall\,\xi \in \mathbb{R}\,, \quad \forall\nu \in \mathcal{V}_1\,,$$

i.e. $B_0 = \delta_1\delta(x-1)$ is Dirac's δ-function concentrated at $x = 1$.
$C_0 : \mathcal{V}_0 \to \mathbb{R}$ *(measurement operator)* is given by

$$C_0\nu := \int_0^1 k(x)\nu(x)dx\ , \quad \forall\nu \in \mathcal{V}_0.$$

Variational solution of (30), (31)

A pair of functions $(\theta(x,t), w(t))$ is a *weak solution* of (27), (28) on $(0,T)$ if

$$\theta(\cdot,t) \in W^{1,2}(0,1)\ , \qquad w, \dot{w} \in L^2(0,T)\ ,$$

$$\int_0^T \Big\{ \int_0^1 [\theta\eta_t - (\delta_1\theta_x\eta_x + \delta_2\theta\eta)]\, dx + \delta_1\delta_3[\phi(t,w) + g(t)]\,\eta\,(1,t)\Big\} dt = 0\ , \tag{32}$$

$$\int_0^T \Big\{ w(t)\zeta(t) + \Big(\int_0^1 \theta(x,t)k(x)\, dx + \delta_4[\phi(t,w) + g(t)]\Big)\,\zeta\,(t)\Big\}\, dt = 0\ , \tag{33}$$

$\forall$ smooth test function $\eta(x,t)$, $\eta(x,0) = \eta(x,1) = 0$,
$\forall$ smooth test function $\zeta(t)$, $\zeta(0) = \zeta(T) = 0$.

(A6):

Transfer function: $\chi(p) = \int_0^1 \tilde{\theta}(x,p)\,dx$ where $\tilde{\theta}(x,p)$ is the solution of the BVP $(k(x) \equiv 1, \delta_3 = 1,\ \delta_4 = -1,\ \delta_5(t) \equiv \delta_5)$:

$$p\,\tilde{\theta} = \delta_1 \tilde{\theta}'' - \delta_2 \tilde{\theta}\ ,$$

$$\tilde{\theta}'_{|_{x=0}} = 0, \quad \tilde{\theta}'_{|_{x=1}} = 1\ .$$

$$\Rightarrow \quad \tilde{\theta}(x,p) = \frac{\cosh \sqrt{p+\delta_2}\,x}{\sqrt{p+\delta_2}\sinh\sqrt{p+\delta_2}}\ ,$$

$$\Rightarrow \quad \chi(p) = \frac{1}{\sqrt{p+\delta_2}\sinh\sqrt{p+\delta_2}} \int_0^1 \cosh\sqrt{p+\delta_2}\,dx = \frac{1}{p+\delta_2}\ ,$$

$\Rightarrow$ sufficient to assume that

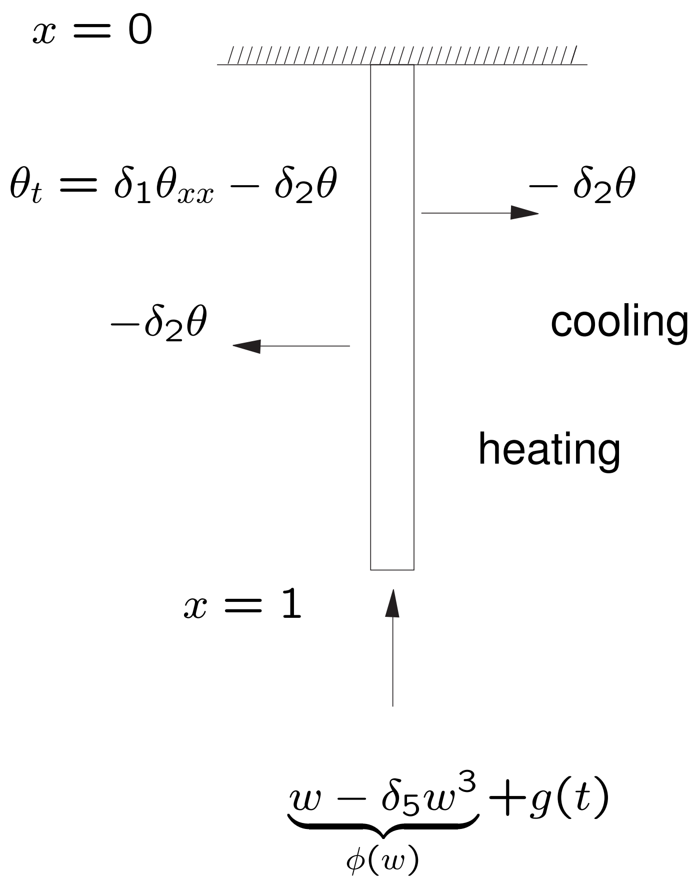
$$\boxed{|g(t)| < \frac{2}{3\sqrt{3\,\delta_5}}} \qquad \text{a.e. } t \in \mathbb{R}\ .$$

$\kappa_2 = \phi'(r_1),\ \kappa_3 = 1 \qquad \Rightarrow$ **(A6)**

$$\chi(p) = \frac{1}{p+\delta_2}$$

$\lambda \in (0, \delta_2) \qquad \Rightarrow$ **(A2)**

$\lambda^2 - \delta_2\lambda + \kappa_1 \le 0 \qquad \Rightarrow$ **(A5)**



$\boxed{\delta_2^2 \geq 4\,\kappa_1} \Rightarrow$ **(A3)** + **(A6)** - - → cooling

condition