Extensions of Lyapunov's ideas in the algebraic approximation of attractors

G. A. Leonov and V. Reitmann[*)]

Saint-Petersburg State University, Russia

Department of Mathematics and Mechanics

[*)]DAAD-Langzeitdozent

International Congress "Nonlinear Dynamical Analysis - 2007"

June 4 - 8, 2007, St. Petersburg, Russia

1 "Universal" dynamical models

a) Continuous time $\mathbb{T} = \mathbb{R}_+$: PDE's of the type

$$\frac{\partial \sigma}{\partial t} + \sum_{j=1}^{n} A_j \frac{\partial \sigma}{\partial x_j} + B\varphi(\sigma, t) = h(x, t),\ (x, t) \in \mathbb{R}^n \times (0, \infty) \tag{1.1}$$

$$\sigma : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m\,,\ \sigma = (\sigma_1, \ldots, \sigma_m)$$

$$\sigma(0, x) = \sigma^0(x)\,,\ x \in \mathbb{R}^n$$

$$A_j : \mathbb{R}^n \times [0, \infty) \to \mathbb{M}^{m\times m}\,,\ j = 1, 2, \ldots, n,\ B \in \mathbb{M}^{m\times p}$$

$$\varphi : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^p\,,\ h : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$$

Transport equation or scattering equation of Boltzmann-type

b) Discrete time $\mathbb{T} = \mathbb{N}_0$: CML's (Coupled maps lattices) of the type

$$\sigma_j(n+1) = a\sigma_j(n) + B\varphi(\sigma_1(n), n) + h_j(n)\,,\ n, j \in \mathbb{N}_0 \tag{1.2}$$

$$\sigma(n) = (\sigma_1(n), \sigma_2(n), \ldots)\,,$$

$\sigma(0) = (\sigma_1^0, \sigma_2^0, \ldots)\,,$ Coupled maps lattices for the transport equation

$$a \in \mathbb{R}\,,\ B \in \mathbb{M}^{1\times p}$$

$$\varphi : \mathbb{R} \times \mathbb{N}_0 \to \mathbb{R}^p$$

2 Realization of a Volterra integral equation as abstract Boltzmann-type transport equation

Consider the nonlinear Volterra integral equation

$$\sigma(t) = h(t) + \int_0^t G(t-s)\varphi(\sigma(s), s)\,\mathrm{d}s \tag{2.1}$$

with $\sigma : \mathbb{R}_+ \to U$ ($\mathbb{R}^n$, Hilbert space) as output,
$h : \mathbb{R}_+ \to U$ as perturbation,
$u(\cdot) := \varphi(\sigma(\cdot), \cdot) : \mathbb{R}_+ \to U$ as control,
$\forall t \geq 0 : G(t) \in \mathcal{L}(U, U)$ as kernel and
$\varphi : U \times \mathbb{R}_+ \to U$ as nonlinearity .

(A1) $t \mapsto G(t)$ is twice piecewise-differentiable and

$$\exists\, c > 0,\ \rho_0 > 0 : \|G(t)\|_{\mathcal{L}(U,U)} \leq c e^{-\rho_0 t}, \quad \forall t \geq 0,$$
$$\int_0^\infty \left(\|\dot{G}(t)\|^2_{\mathcal{L}(U,U)} + \|\ddot{G}(t)\|^2_{\mathcal{L}(U,U)} \right) e^{2\rho_0 t}\,\mathrm{d}t < \infty\,.$$

(A2) $\exists\, P = P^*, Q, R \in \mathcal{L}(U, U)$,

$$(\sigma(t), P\sigma(t))_U + 2(\sigma(t), Q\varphi(\sigma(t), t))_U + (\varphi(\sigma(t), t), R\varphi(\sigma(t), t))_U \leq 0 \tag{2.2}$$

$\forall \sigma(\cdot), \varphi(\sigma(\cdot), \cdot)\,, \sigma(\cdot)$ continuous solution from (2.1)

(Quadratic constraints)

$$L^2_\rho(\mathbb{R}_+; U) := \left\{ f \in L^2_{loc}(\mathbb{R}_+; U) : \int_0^\infty |f(t)|^2_U\, e^{2\rho t}\,\mathrm{d}t < \infty \right\}$$

is a weighted L^2-space ,

$$W^{1,2}_\rho(\mathbb{R}_+; U) := \left\{ f \in L^2_\rho(\mathbb{R}_+; U) : \dot{f} \in L^2_\rho(\mathbb{R}_+; U) \right\}$$

is a weighted Sobolev space .

(A3) The linear part of (2.1) is $\underline{\rho\text{-stable}}$, i.e.

$$\exists\, \rho \geq 0 \;\forall u \in L^2(\mathbb{R}_+; U) \mapsto \sigma(\cdot) \in W^{1,2}_{\rho}(\mathbb{R}_+; U)\,,$$

$$\sigma(t) = \int_0^t G(t-s)u(s)\,\mathrm{d}s$$

is a bounded operator.

Goal: Find Hilbert spaces $Z_1 \subset Z_0 \subset Z_{-1}$ (Rigged Hilbert space structure) and linear bounded operators

$$A : Z_1 \to Z_{-1},\; B : U \to Z_{-1},\; C : Z_0 \to U$$

such that the global stability behaviour of (2.1) coincides with the global stability behaviour of the non-autonomous dynamical system

$$\begin{aligned} \dot{z} &= Az + Bu(t) \\ \sigma &= Cz,\; u(t) = \varphi(\sigma(t), t)\,, \end{aligned} \tag{2.3}$$

and the following conditions are satisfied:

(i) $z(0, z_0, u) = z_0 \quad \forall z_0 \in Z_0, \quad \forall u \in L^2_{loc}(0, \infty; U)\,,$
(Initial condition);

(ii) $u(t) = 0, \quad \forall t \leq T \;\Rightarrow\; z(t, 0, u) = 0, \quad \sigma(t, 0, u) = 0, \quad \forall t \leq T,$
(Causality);

(iii) $z(t+s, z_0, u) = z(t, z(s, z_0, u), \tau^s u)$
$\sigma(t+s, z_0, u) = \sigma(t, z(s, z_0, u), \tau^s u)$
$\forall z_0 \in Z_0,\; \forall u \in L^2_{loc}(0, \infty; U), \forall t, s \geq 0,$
(Time-invariance or cocycle property)
with $\tau^s u(t) := \begin{cases} u(t+s) & \text{for} \quad t+s \geq 0\,, \\ 0 & \text{for} \quad t+s < 0 \end{cases}$ as shift semi-group.

Example 2.1

$$\dot{x} = d(y-x)\,, \quad \dot{y} = rx - y - xz\,, \quad \dot{z} = -bz + xy$$

$$d > 0, \quad r > 0, \quad b > 0 \qquad\qquad \underline{\text{Lorenz equation}}$$

Nonlinearities: $\varphi_1 = xz, \quad \varphi_2 = xy, \quad \varphi = (\varphi_1, \varphi_2)\,,\ \sigma = (x, y, z)$

Quadratic constraints: $F(\varphi, \sigma) := \varphi_1 y - \varphi_2 z,$

$$F((xz, xy), \sigma) = xyz - xyz \equiv 0 \quad \hat{=}(2.2)$$

Transfer functions: $\tilde{y} = -W_1(i\omega)\tilde{\varphi}_1$

$\tilde{z} = -W_2(i\omega)\tilde{\varphi}_2$

$$\left.\begin{array}{l} W_1(p) = \frac{p+d}{p^2+p(d+1)+d(1-r)} \\ W_2(p) = -\frac{1}{p+b} \end{array}\right\} \Rightarrow G(t)$$

Frequency domain condition (2.5) is satisfied for $0 < r < 1$

Associated Boltzmann equation in $W^{1,2}_{\rho}(\mathbb{R}_+;\mathbb{R}^3)$

□

[7] Foias, C. and R. Temam, *The algebraic approximation of attractors; The finite dimensional case.* Phys. D, **32**, 163 - 182 (1988).

[8] Foias, C., Jolly, M. S., and I. Kukavica, *Localization of attractors by their analytic properties.* Nonlinearity, **9** 1565 - 1581 (1996).

[9] Foias, C., Jolly, M. S., Kukavica, I., and E. S. Titi, *The Lorenz equation as a methaphor for the Navier-Stokes equations.* Discrete and Cont. Dyn. Systems, **7** (2), 403 - 429 (2001).

[19] Leonov, G. A., *On a method of construction of positively invariant sets for the Lorenz system.* Priklad. Mat. Mekh., **49** (5), 860 - 863 (1985). (Russian)

[20] Leonov, G. A., *On the asymptotic behavior of solutions of the Lorenz system.* Differents. Uravn., **25**, 2103 - 2109 (1989). (Russian)

[21] Leonov, G. A., Bunin, A. I., and N. Koksch, *Localization of the attractor of the Lorenz system.* Z. Angew. Math. Mech., **67**, 649 - 656 (1987).

[27] Starkov, K. E. and A. P. Krishchenko, *Localization of polynomial systems by ellipsoidal estimates chaos.* Solitons & Fractals, **23**, 981 - 988 (2005).

Consider the equation

$$u' + Au + R(u) = 0\,,$$

where A is a symmetric positive definite operator in $\mathbb{R}^d$ and
$R(u) = R_0 + R_1(u) + R_2(u)$.
$R_0 \in \mathbb{R}^d$, R_1 is a linear operator in $\mathbb{R}^d$ and $R_2(u) = R_2(u,u)$, where $R_2(\cdot,\cdot)$ is a bilinear operator in $\mathbb{R}^d$.
By integration of the equation between $t < 0$ and 0 one finds

$$u(0) = e^{tA}u(t) - \int_t^0 e^{\tau A} R(u(\tau))\, d\tau.$$

Letting $t \to -\infty$ and remembering that $u(\cdot)$ is bounded, one finds an integral equation.

Example 2.2 Bloch equations

$$\begin{aligned}
\dot{x} &= -\beta x + y\,, \\
\dot{y} &= -x - \beta y\,(1 - kz)\,, \\
\dot{z} &= \beta\,[\alpha(1 - z) - ky^2]\,,
\end{aligned}$$

where $\beta \geq 0, \alpha \geq 0$ and k are parameters. □

[3] Borg, G., *A condition for the existence of orbitally stable solutions of dynamical systems.* Kungl. Tekn. Hogsk. Handl., Stockholm **153**, 1623 (1960).

[18] Leonov, G. A., *An analog of Bendixson's criterion for third order equations.* Diff. Eqns. **13** (2), 367 - 368 (1977). (Russian)

[26] Sherman, S., *A third order nonlinear system arising from nuclear spin generator.* Contr. Diff. Eqns., 197 - 227 (1963).

Theorem 2.1. *(Realization theorem of Helton ([11]), Kalman ([12]) Salamon ([25]))*
Suppose that (2.1) is linear and the input / output process given by (2.1) is ρ-stable. Then there exists an imbedding of (2.1) into a system (2.3) with the same global stability behaviour by the <u>*Boltzmann-type transport equation*</u>*, i.e. by a system (2.3)*

$$\text{with} \quad Z_0 := W_\rho^{1,2}(\mathbb{R}_+;U),\ 0<\rho<\rho_0$$

$$Z_1 := D(A) = \{\xi : \xi(s) \in W_\rho^{1,2}(\mathbb{R}_+;U), \int_0^\infty e^{2\rho s}|\ddot{\xi}(s)|^2 \mathrm{d}s < \infty\},$$

$$(A\,\xi)(s) := \frac{\partial \xi(s)}{\partial s} \ \text{transport or impulse operator} \quad (2.4)$$

$$(B\eta)(s) := G(s)\eta,\ \eta \in U(=\mathbb{R}^n),\ Cz(s) := z(0),\ \forall z(s) \in W_\rho^{1,2}(\mathbb{R}_+;U).$$

[11] Helton, J. W., *Systems with infinite-dimensional state space: the Hilbert space approach.* Proc. IEEE, **64**, 145–160 (1976).

[12] Kalman, R.E., Arbib, M., and P. Falb, Topics in Mathematical Systems Theory. McGraw-Hill, New York (1969).

[15] Lax, P.D. and R. S. Phillips, *Scattering theory for dissipative hyperbolic systems.* J. Funct. Anal., **14**, 172 – 235 (1973).

[25] Salamon, D., *Realization Theory in Hilbert Space,* Math. System Theory. **21**, 147–164 (1989).

[28] Steil, J. J. and H. Ritter, *Input-output stability of recurrent neural networks with delays using circle criteria.* In Proc. ICSC / IFAC Symp. NEURAL COMPUTATION, NC' 98, ICSC Academic Press, 519–225 (1998).

[31] Yakubovich, V.A., *On the abstract theory of absolute stability of nonlinear systems.* Vestn. Leningr. Univ., Ser. Mat. Mekh., Astran., **13**, 99–118 (1977).

Example 2.3 Consider

$$\begin{aligned}
&\dot{y} = Ay + B\varphi(\sigma(t), t), \sigma(t) = Cy(t) \\
&A - n \times n,\ B - n \times 1,\ C - 1 \times n \quad \text{matrices} \\
&\varphi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \\
&G(t) := Ce^{At}B\,, \quad h(t) = Ce^{At}y_0 \\
&\sigma(t) = h(t) + \int_0^t G(t-s)\,\varphi\,(\sigma(s), s)\mathrm{d}s
\end{aligned}$$

The nonlinear Boltzmann transport equation from scattering theory

$$\frac{\partial \sigma}{\partial t} = \frac{\partial \sigma}{\partial x} + \int_0^x G(x-s)\varphi(\sigma(s,x))\,\mathrm{d}s$$

is a first order integro-differential equation with boundary and initial conditions

$$\sigma(t,0) = 0, \quad \sigma(0,x) = \sigma_0(x)$$

□

Suppose that we know G, P, Q and R. Then the abstract evolution system gives the following information:

Theorem 2.2. *(Generalized Brusin's theorem) Consider the nonlinear Volterra integral equation (2.1) under the assumptions* **(A1)** *–* **(A3)**. *Let* $\hat{G}(\lambda) := \int_0^\infty e^{-\lambda t} G(t)\,\mathrm{d}t$ *be the transfer operator of the kernel. Assume that the class of nonlinearities described by* **(A2)** *contains at least one linear function* $\varphi(\sigma, t) = K\sigma$ *with* $K \in \mathcal{L}(U, U)$ *such that the operator* $(I - \hat{G}(\lambda)K)^{-1}$ *has a finite number of singularities in the strip* $0 < \varepsilon_1 \leq Re\,\lambda \leq \varepsilon_2$.
Suppose that the <u>*frequency–domain condition*</u>

$$\hat{G}^*(i\omega)P\hat{G}(i\omega) + 2Re(Q^*\hat{G}(i\omega)) + R > 0 \quad \forall \omega \in \mathbb{R} \tag{2.5}$$

is satisfied.
Then the nonlinear integral equation (2.1) can be imbedded into a non-autonomous dynamical system (2.3) with the same global stability behaviour realized as transport equation (2.4), i.e. there exists linear bounded operator $M = M^* : W_\rho^{1,2}(\mathbb{R}_+; U) \to W_\rho^{1,2}(\mathbb{R}_+; U)$ *with the following properties.*
1) If $\sigma(t) = \sigma(t, h)$ *with* $h \in W_\rho^{1,2}(\mathbb{R}_+; U)$ *is a continuous solution of the integral equation (2.1) then the solution* $z(t) = z(t, h)$ *of (2.3) with* $z(0, h) = h$ *exists and there is a positive* $\delta > 0$ *such that*

$$\int_{t_1}^{t_2} \left(|\, \varphi(\sigma(t), t)\,|_U^2 + \|\, z(t)\,\|_{W_\rho^{1,2}}^2 \right) \mathrm{d}t \leq \delta \,\left(Mz(t), z(t) \right)\Big|_{t_1}^{t_2}$$

$$\forall\, 0 \leq t_1 < t_2$$

2) Suppose $(\sigma(\cdot), h(\cdot))$ *satisfies (2.1) and* $(h(\cdot), Mh(\cdot)) < 0$. *Then* $\sigma(\cdot) \in L^2(\mathbb{R}_+; U)$, *i.e. it is stable. If* $(h(\cdot), Mh(\cdot)) > 0$ *then* $\sigma(\cdot)$ *is unstable, i.e. there exists a number* $\beta > 0$ *such that*

$$\lim_{T\to\infty} e^{-\beta T} \int_0^T |\, \varphi(\sigma(t), t)\,|_U^2 \, \mathrm{d}t = \infty \, .$$

3) M is the operator solution of a <u>*linear*</u> *integral equation.*

[4] Brusin, V. A., *On a class of nonautonomous dynamical systems defined by nonlinear integral equations.* Izv. Vyssh. Uchevn. Zaved., Radiofizika, **13** 4, 491–500 (1970).

[5] Brusin, V. A., *Use of method of abstract differential equations in the study of Volterra integral equation.* Sibirskii Mathematicheskii Zhurnal, **18**, 1246–1258 (1977).

[24] Reitmann, V. and H. Kantz, *Stability investigation of Volterra integral equations by realization theory and frequency-domain methods.* Preprint-Series DFG-SPP 1114, Preprint 61 (2004). (www.math.uni-bremen.de/zetem/DFG-Schwerpunkt)

Remark 2.1 The (algebraic) dimension d of the cone $\{h \in W_\rho^{1,2}(\mathbb{R}_+; U) \; : \; (Mh, h) > 0\}$ is finite and coincides with the topological dimension of an orbit closure $\mathfrak{M} := cl\{z(t), t \geq 0\}$ of system (2.3). Thus d real coordinates are sufficient to describe by a one-to-one map the points in $\mathfrak{M}$. □

[2] Boichenko, V. A., Leonov, G. A., and V. Reitmann, Dimension Theory for Ordinary Differential Equation. Teubner, Stuttgart (2005)

3 Realization of a time-series as output of a discrete-time Boltzmann-type transport equation

Consider the nonlinear system

$$\sigma(k) = h(k) + \sum_{j=0}^{k-1} G(k-j-1)\varphi\left(\sigma(j), j\right), \tag{3.1}$$

$k = 1, 2, \ldots,\ \sigma(0) = \sigma_0,\ \{\sigma(k)\}_{k=1}^{\infty}$ time-series generated by (3.1)

$h : \mathbb{N}_0 \to U\ ,\ U$: Hilbert space, $\mathbb{R}^n$

$G(j) \in \mathcal{L}(U, U)$

$\varphi : U \times \mathbb{N}_0 \to U$

$\hat{G}(p) := \sum_{k=0}^{\infty} G(k)p^{-k}$ z-transform of G,

$T(p) := \hat{G}(1/p) = \hat{G}_0 + \hat{G}_1 p + \hat{G}_2 p^2 + \ldots$

($\tilde{\mathbf{A}}$1) $\exists\, c > 0 \quad \exists\, \rho_0 > 1\ :\ \|\, G(k)\,\|_{\mathcal{L}(U,U)} \le c\rho_0^{-k}\ ,\ k = 1, 2, \ldots.$

($\tilde{\mathbf{A}}$2) $\exists P = P^*, Q, R \in \mathcal{L}(U, U)$ s.t.

$(\sigma(k), P\sigma(k))_U + 2(\sigma(k), Q\varphi(\sigma(k), k))_U$
$+ (\varphi(\sigma(k), k),\ R\varphi(\sigma(k), k))_U \le 0$
$\forall \{\sigma(k)\}$ solution of (3.1), $\{\varphi(\sigma(k), k)\}$.

Let $\ell^2_\rho(1, \infty; U)$ with $0 < \rho < \infty$ be the set of sequences $u = (u_1, u_2, \ldots)$ with $u_k \in U$ for which $\{\rho^{-k} u_k\}$ belongs to $\ell^2(1, \infty; U)$, i.e. $\sum_{k=1}^{\infty} \rho^{-2k}\, \|u_k\|_U^2 < \infty$.

($\tilde{\mathbf{A}}$3) The linear part of (3.1) is ρ-stable, i.e. $\exists\, \rho > 0 \quad \forall u \in \ell^2(1, \infty; U)$ the sequence $\{\sigma(k)\}, \sigma(k) := \sum_{j=0}^{k-1} G(k-j-1)u(j)$ belongs to $\ell^2_\rho(1, \infty; U)$.

Realization procedure:

1) Introduce the backward shift $\tau : \ell^2_\rho(1,\infty;U) \to \ell^2_\rho(1,\infty;U)$ by
$\tau(u_1,u_2,\ldots) = (u_2,u_3,\ldots),\ \forall u = (u_1,u_2,\ldots) \in \ell^2_\rho(1,\infty;U)$.
Define $A := \tau$

2) Define $B : U \to \ell^2_\rho(1,\infty;U)$ by

$$Bu := (\hat{G}_1 u, \hat{G}_2 u, \ldots),\ \forall u \in U\,,$$

and $C : \ell^2_\rho(1,\infty;U) \to U$ by

$$C(u_1,u_2,\ldots) := u_1\,,\ \forall (u_1,u_2,\ldots) \in \ell^2_\rho(1,\infty;U).$$

3)
$$\begin{aligned} z(k+1) &= Az(k) + Bu(k)\,, \\ \sigma(k) &= Cz(k), z(0) = z_0 \in \ell^2_\rho(1,\infty;U) \\ u(k) &= \varphi(\sigma(k),k)\,,\ k = 0,1,2,\ldots\,. \end{aligned} \tag{3.2}$$

(3.2) is called discrete-time Boltzmann-type transport equation associated with (3.1).

Example 3.1 $\sigma_{n+2} + \sigma_{n+1} + \varphi(\sigma_n, n) = 0\,, \quad n = 0,1,2,\ldots$

$\sigma(0) = \sigma_0, \quad \sigma(1) = \sigma_1$

z-transform: $p^2\tilde{\sigma} + p\tilde{\sigma} = -\tilde{\varphi}$

$\hat{G}(p) = \frac{1}{p^2+p}$

$$\hat{G}\Big(\frac{1}{p}\Big) = \frac{1}{\frac{1}{p^2}+\frac{1}{p}} = \frac{p^2}{1+p} = p - 1 + \frac{1}{1+p} = p - 1 + \sum_{m=0}^{\infty}(-1)^m p^m$$
$$= p^2 - p^3 + p^4 - \cdots$$

$$\hat{G}_m = \begin{cases} 0\,, & m = 0, 1\,, \\ (-1)^m\,, & m = 2, 3, \ldots \end{cases}$$

$$\begin{aligned} &(z_1(k+1), z_2(k+2), \ldots) \\ &= \tau(z_1(k), z_2(k), \ldots) + (0, \varphi(\sigma_k, k), -\varphi(\sigma_k, k), \ldots)\,, \\ &k = 0, 1, 2, \ldots\,, \quad \sigma_k = z_1(k)\,, \end{aligned}$$

$$z_m(k+1) = z_{m+1}(k) + (-1)^m \varphi(z_1(k), k), m, k \in \mathbb{N}_0$$

□

Space- and time-discrete version of the Ginsburg-Landau equation in $\ell^2_\rho(1, \infty; U)$:

$$\begin{aligned} u_j(n+1) &= u_j(n) - (1 - i\beta) u_j(n) \,|\, u_j(n) \,|^2 \\ &+ \varkappa(u_{j-1}(n) - 2u_j(n) + u_{j+1}(n)),\ u_j(n) \in \mathbb{C},\ n, j \in \mathbb{Z}. \end{aligned}$$

Dynamical objects of the lattice model associated to a time-series:

- finite-dimensional attractors
- hyperbolicity
- travelling waves $\quad u_j(n) = \Psi(l_j + mn)$
- spatial structures

Theorem 3.1. *Consider the iteration (3.1) under the assumptions* $(\tilde{\mathbf{A}}\mathbf{1})$ – $(\tilde{\mathbf{A}}\mathbf{3})$. *Let* $\hat{G}(p) := \sum_{k=0}^{\infty} G(k)p^{-k}$ *be the z-transform of G. Assume that the class of nonlinearities described by* $(\tilde{\mathbf{A}}\mathbf{2})$ *contains at least one linear function* $\varphi(\sigma, t) = K\sigma$ *with* $K \in \mathcal{L}(U, U)$ *such that the operator* $(I - \hat{G}(p)K)^{-1}$ *has a finite number of singularities in the ring*

$$1 < \varepsilon_1 \leq |\, p \,| \leq \varepsilon_2.$$

Suppose that the frequency-domain condition

$$\hat{G}^*(p)P\hat{G}^*(p) + 2Re\,(Q^*\hat{G}(p)) + R > 0, \quad \forall p \in \mathbb{C} : |\, p \,| = 1$$

is satisfied.
Then there exists a linear bounded operator $M = M^* : \ell^2_\rho(1, \infty; U) \to \ell^2_\rho(1, \infty; U)$ *with the following property: Suppose* $\sigma = \{\sigma(k)\}_{k=1}^{\infty}$ *is a sequence generated by (3.1) with* $h = \{h(k)\}_{k=1}^{\infty}$. *Then if* $(h, Mh) < 0$ *we have* $\sigma \in \ell^2_\rho(1, \infty; U)$, *i.e.* σ *is stable. If* $(h, Mh) > 0$ *then* σ *is unstable.*

Remark 3.1 $z(k) = \tau^k z(0) + \sum_{j=0}^{k-1} \tau^{k-j-1} Bu(j),\ k = 1, 2, \ldots,$
$z(0) = (z_0(0), z_1(0), \ldots)$ a time-series , $\mathfrak{M}(z(0)) := cl(\{\tau^k(z(0))\}_{k=0}^{\infty})$ the orbit closure. Suppose $\dim_F \mathfrak{M} =: d < \infty$ and let n be the smallest natural number s.t. $n \geq 2d + 1$.
$M := \{(z_0, z_1, \ldots, z_{n-1})\}$ is an n-dimensional subspace of $\ell^2_\rho(1, \infty; U)$.
The typical projections $\ell^2_\rho(1, \infty; U) \to M$ are one-to-one. Let the standard projection π_n be typical. Then on
$E := \pi_n(\mathfrak{M}(z(0))$ there is given a dynamical system $\tilde{\tau} := \pi_n \circ \tau \circ \pi_n^{-1}$

$$(\tilde{\tau}, M) : (z_0, z_1, \ldots, z_{n-1}) \mapsto (z_1, z_2, \ldots, z_n).$$

□

4 Transport equation for the Mathieu-Hill equation (in cooperation with N. Kuznetsov)

We consider an ODE of the second order

$$\ddot{\sigma} + \alpha\dot{\sigma} + \varphi(\sigma(t), t) = 0 \tag{4.1}$$

with a smooth nonlinearity $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Assume that any solution of (4.1) exists on $\mathbb{R}$.

Let us rewrite (4.1) in the following way

$$\begin{cases} \dot{z}(t) = Az(t) + B\varphi(\sigma(t), t) \\ \sigma(t) = Cz(t)\,, \end{cases} \tag{4.2}$$

$$\text{with} \quad A = \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where $\sigma(t)$ is the input and $\varphi(\sigma(t), t)$ is the output.

As "nonlinear part" is considered the function

$$\varphi(\sigma, t) = (\beta + \gamma\cos(t))\sigma\,, \tag{4.3}$$

where β and γ are parameters. Note that equation (4.1) with φ given by (4.3) has the form of the Mathieu-Hill equation. It is well-known ([13, 17, 14]) that this equation with parametric exicitation can be used to describe some bifurcations in dynamical buckling processes.

Time is considered on the finite interval $[0, T]$.

All functions are considered as sequences

$$\{\sigma(t_i)\}_1^{N+1}, \; t_k = (k-1)\frac{T}{N}, k = 1, 2.., N+1$$

where $N+1$ is the number of nodes on the interval $[0, T]$.

Step 1

Find a sector for the nonlinear part such that

$\mu_1 \leq \varphi(\sigma, t)/\sigma \leq \mu_2 \quad \forall t, \sigma.$

Take initial data $(\sigma_i(0), \dot{\sigma}_i(0)),\ i = 1, 2.., L$, and calculate the numbers μ_1, μ_2 such that the relation

$\mu_1 \leq \varphi(\sigma(t_i), t_i)/\sigma(t_i) \leq \mu_2\,, \quad i = 1, .., N+1\,,$

is satisfied. For the calculation of μ_1, μ_2 an adaptive algorithm is used which is finitely converging in the sense of Yakubovich ([10]).

Step 2

Write system (4.1) as Volterra integral equation

$$\begin{aligned} \sigma(t) &= h(t) + \int_0^t G(t-\tau)\varphi(\sigma(\tau), \tau)\,\mathrm{d}\tau\,, \\ \varphi(\sigma, t) &= (\beta + \gamma\cos(t))\sigma, \end{aligned} \tag{4.4}$$

where $h(\cdot)$ is the input and $\sigma(\cdot)$ the output ($\sigma \equiv \sigma_h$).

The goal is to construct an operator M which gives all information about stability of $\sigma_h(\cdot)$ with respect to the input $h(\cdot)$.

Assume that the kernel of (4.4) can be written as

$$G(t-\tau) = e^{\lambda(t-\tau)},$$

where λ is an unknown parameter.

Let $\rho \geq 0$ be the unknown parameter of the Hilbert space L^2_ρ introduced in Section 2.

Step 3

In order to construct the operator M we have as an auxiliary problem to solve the linear Fredholm integral equation of the second kind

$$\int_0^T S_{(\rho,\lambda)}(t,\tau)\tilde{u}_{h,(\rho,\lambda)}(\tau)\mathrm{d}\tau + \tilde{u}_{h,(\rho,\lambda)}(t) = g_{h,(\rho,\lambda)}(t), \tag{4.5}$$

where $S_{(\rho,\lambda)}$ is a function depending on ρ and λ, and $g_{h,(\rho,\lambda)}$ depends also on $h(\cdot)$.

From this equation we get $\tilde{u}_{h,(\rho,\lambda)}(\cdot)$ which will be used further.

Remark 1

If we solve the integral equation (4.5) we get the solution of an associated Riccati equation. In general the Riccati equation is a quadratic equation with respect to the unknown matrix or operator. In our situation this equation (4.5) is linear what is important for practical realization. The reason for this is the special type of hyperbolic equations arising in (3). This property was also investigated in [30].

Step 4

Construct the cost-functional $J^T_{\lambda,\rho}(\cdot)$ on L^2_ρ.

Take some initial values $\overline{\lambda}, \overline{\rho}$, calculate the functional with these parameters and compare with the data.

Use for this an optimization procedure with respect to λ, ρ for the functional computed along the solution of the Fredholm integral equation (4.5).

As result of this step we get the functional $J^T_{\lambda_0,\rho_0}$.

Step 5

Define the operator M^T by

$$(M^T h)(s) := -\frac{1}{\lambda_1}\int_0^T \{e^{-2\rho(\tau)}\left[e_{s-\tau}e^{\lambda_1(s-\tau)} + e_{\tau-s}\mu_1(s-\tau)\right] +$$

$$+\mu_2(-\tau)e^{\lambda_1 s}\}(P\widetilde{\sigma}_h(\tau) + Qh(\tau))\,\mathrm{d}\tau,\ \forall h \in W^{1,2}_{\rho_0}, \qquad (4.6)$$

where $\widetilde{\sigma}_h(t) = \int_0^t (e^{\lambda_0(t-\tau)} + h(\tau))\mathrm{d}\tau + h(t)$ and the functions $\lambda_1(\cdot), \mu_1(\cdot), \mu_2(\cdot)$ depend only on ρ_0.

Then the sign of the test functional

$$< M^T h, h >= \int_0^T (M^T h)(s)h(s)e^{2\rho_0 s}\,\mathrm{ds} \qquad (4.7)$$

gives us the information about stability of $\sigma(\cdot)$ according to Brusin's theorem (Theorem 2).

5 Numerical results

Consider the equation (4.1),(4.2) with the system parameters

$$\alpha = 1/3;\ \beta = 1;\ \gamma = 2. \tag{5.1}$$

Using the above algorithm with $T = 2\pi, N = 18, L = 50$ we find the sector from Step 1 for the "nonlinearity" (4.3) with $\mu_1 = -1, \mu_2 = 3$.
For the kernel $G(\cdot)$ and the function space L^2_ρ we obtain the parameters

$$\lambda_0 = 0.29,\ \rho_0 = 0.1\ . \tag{5.2}$$

This defines the operator M^T for the test functional (4.6)
In order to verify our result we consider the solution of (4.4) with the initial data

$$\sigma(0) = 0.15683, \dot{\sigma}(0) = 0, 25269\ . \tag{5.3}$$

Computing the associated h in (4.4) we get a positive sign of the test functional (4.6). According to Brusin's theorem the solution must be unstable. The direct calculation of the solution (Fig. 3) shows their instability.
This means that the information from test functional (4.7) is correct.

References

[1] Banks, H. T., Fabiano, R. H., and Y. Wang, *Inverse problem techniques for beams with tip body and time hysteresis damping.* Mat. Aplic. Comp., **8** 2, 101–118 (1989).

[2] Boichenko, V. A., Leonov, G. A., and V. Reitmann, Dimension Theory for Ordinary Differential Equations. Teubner, Stuttgart (2005).

[3] Borg, G., *A condition for the existence of orbitally stable solutions of dynamical systems.* Kungl. Tekn. Hogsk. Handl., Stockholm **153**, 1623 (1960)

[4] Brusin, V. A., *On a class of nonautonomous dynamical systems defined by nonlinear integral equations.* Izv. Vyssh. Uchevn. Zaved., Radiofizika, **13** 4, 491–500 (1970).

[5] Brusin, V. A., *Use of method of abstract differential equations in the study of Volterra integral equation.* Sibirskii Mathematicheskii Zhurnal, **18**, 1246–1258 (1977).

[6] Cavaterra, C., *An inverse problem for a viscoelastic Timoshenko beam model.* ZAA, **17** 1, 67–87 (1998).

[7] Foias, C. and R. Temam, *The algebraic approximation of attractors; The finite dimensional case.* Phys. D, **32**, 163 - 182 (1988).

[8] Foias, C., Jolly, M. S., and I. Kukavica, *Localization of attractors by their analytic properties.* Nonlinearity, **9** 1565 - 1581 (1996).

[9] Foias, C., Jolly, M. S., Kukavica, I., and E. S. Titi, *The Lorenz equation as a methaphor for the Navier-Stokes equations.* Discrete and Cont. Dyn. Systems, **7** (2), 403 - 429 (2001).

[10] Fomin, V. N., Fradkov, A. L., and V. A. Yakubovich, Adaptive Control of Dynamical Objects. Moscow, Nauka, (1981). (in Russian)

[11] Helton, J. W., *Systems with infinite-dimensional state space: the Hilbert space approach.* Proc. IEEE, **64**, 145–160 (1976).

[12] Kalman, R.E., Arbib, M., and P. Falb, Topics in Mathematical Systems Theory. McGraw-Hill, New York (1969).

[13] Karagiozova, D. and N. Jones, *Strain-rate effects in the dynamic buckling of a simple elastic-plastic model.* J. Appl. Mechanics, **64**, 193 – 200 (1997).

[14] Kirkpatrick, S. W., Schroeder, M., and J. W. Simons, *Evaluation of passenger rail vehichle crash worthiness.* Intern. J. of Crashworthiness, **6** 1, 95–106 (2001).

[15] Lax, P.D. and R. S. Phillips, Scattering Theory. Academic Press, New York (1967).

[16] Lax, P.D. and R. S. Phillips, *Scattering theory for dissipative hyperbolic systems.* J. Funct. Anal., **14**, 172 – 235 (1973).

[17] Lee, L. H. N., *Quasi-bifurcation of rods within an axial plastic compressive wave.* Trans. ASME, **45**, 100–104 (1978).

[18] Leonov, G. A., *An analog of Bendixson's criterion for third order equations.* Diff. Eqns. **13** (2), 367 - 368 (1977). (Russian)

[19] Leonov, G. A., *On a method of construction of positively invariant sets for the Lorenz system.* Priklad. Mat. Mekh., **49** (5), 860 - 863 (1985). (Russian)

[20] Leonov, G. A., *On the asymptotic behavior of solutions of the Lorenz system.* Differents. Uravn., **25**, 2103 - 2109 (1989). (Russian)

[21] Leonov, G. A., Bunin, A. I., and N. Koksch, *Localization of the attractor of the Lorenz system.* Z. Angew. Math. Mech., **67**, 649 - 656 (1987).

[22] Osokin, A. E. and Iu. V. Suvorova, *Nonlinear governing equation of a hereditary medium and methodology of determining its parameters.* PMM USSR, **42** 6, 1107–1114 (1978).

[23] Reed, M. and B. Simon, Methods of Modern Mathematical Physics III: Scattering Theory. Academic Press, New York (1979).

[24] Reitmann, V. and H. Kantz, *Stability investigation of Volterra integral equations by realization theory and frequency-domain methods.* Preprint-Series DFG-SPP 1114, Preprint 61 (2004). www.math.uni-bremen.de/zetem/DFG-Schwerpunkt

[25] Salamon, D., *Realization Theory in Hilbert Space,* Math. System Theory. **21**, 147–164 (1989).

[26] Sherman, S., *A third order nonlinear systems arising from nuclear spin generator.* Contr. Diff. Eqns., 197 - 227 (1963).

[27] Starkov, K. E. and A. P. Krishchenko, *Localization of polynomial systems by ellipsoidal estimates chaos*. Solitons & Fractals, **23**, 981 - 988 (2005).

[28] Steil, J. J. and H. Ritter, *Input-output stability of recurrent neural networks with delays using circle criteria.* In Proc. ICSC / IFAC Symp. NEURAL COMPUTATION, NC' 98, ICSC Academic Press, 519–225 (1998).

[29] Tobias, T.A. and Yu. K. Engelbrekht, *Inverse problems for evolution equations of the integrodifferential type.* PMM USSR, **49** 4, 519–524 (1985).

[30] Triggiani, R., *An optimal control problem with unbounded control operator and unbounded observation operator where the algebraic Riccati equation is satisfied as a Lyapunov equation.* Appl. Math. Lett., **10** 2, 95–162 (1997).

[31] Yakubovich, V.A., *On the abstract theory of absolute stability of nonlinear systems.* Vestn. Leningr. Univ., Ser. Mat. Mekh., Astran., **13**, 99–118 (1977).