Determining functionals for bifurcations on a finite-time interval in variational inequalities

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1. Evolutionary variational inequalities

Suppose that Z_0 is a real Hilbert space with $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ as scalar product resp. norm. Suppose also that $A : \mathcal{D}(A) \subset Z_0 \to Z_0$ is an unbounded densely defined linear operator. The Hilbert space Z_1 is defined as $\mathcal{D}(A)$ equipped with the scalar product

$$(z,\eta)_1 := ((\beta I - A)z,\, (\beta I - A)\eta)_0\ , \quad z,\eta \in \mathcal{D}(A)\,, \tag{1}$$

where $\beta \in \rho(A) \cap \mathbb{R}$ ($\rho(A)$ the resolvent set of A) is an arbitrary but fixed number the existence of which we assume.

The Hilbert space Z_{-1} is by definition the completion of Z_0 with respect to the norm $\|z\|_{-1} := \|(\beta I - A)^{-1} z\|_0$. Thus we have the dense and continuous imbedding

$$Z_1 \subset Z_0 \subset Z_{-1} \tag{2}$$

which is called *Hilbert space rigging structure* (Yu. M. Berezanskii, 1965). The *duality pairing* $(\cdot,\cdot)_{-1,1}$ on $Z_{-1} \times Z_1$ is the unique extension by continuity of the functionals $(\cdot, z)_0$ with $z \in Z_1$ onto Z_{-1}. If $T > 0$ is an arbitrary number we define the norm for Bochner measurable functions in $L^2(0,T;Z_j),\ j = 1, 0, -1$ through

$$\|z\|_{2,j} := \left(\int_0^T \|z(t)\|_j^2 \, dt \right)^{1/2} . \tag{3}$$

Let $\mathcal{W}_T$ be the space of functions $z(\cdot) \in L^2(0,T;Z_1)$ for which $\dot{z}(\cdot) \in L^2(0,T;Z_{-1})$, equipped with the norm

$$\|z(\cdot)\|_{\mathcal{W}_T} := \left(\|z(\cdot)\|_{2,1}^2 + \|\dot{z}(\cdot)\|_{2,-1}^2 \right)^{1/2} . \tag{4}$$

Assume that U and W are two other real Hilbert spaces with scalar products $(\cdot,\cdot)_U,\ (\cdot,\cdot)_W$ and norms $\|\cdot\|_U, \|\cdot\|_W$, respectively.

Introduce the linear continuous operators

$$A : Z_1 \to Z_{-1}\ ,\ B : U \to Z_{-1}\ ,\ C : Z_1 \to W\ , \tag{5}$$

(A from above) and define the set-valued map

$$\varphi : \mathbb{R}_+ \times W \to 2^U \tag{6}$$

and the map

$$\psi : Z_1 \to \mathbb{R}_+ \,. \tag{7}$$

Consider the *evolutionary variational inequality* with set-valued non-linearity

$$(\dot z - Az - Bu, \eta - z)_{-1,1} + \psi(\eta) - \psi(z) \geq 0\,, \quad \forall\, \eta \in Z_1\,, \tag{8}$$

$$w(t) = Cz(t)\ ,\ u(t) \in \varphi(t, w(t))\ ,\ y(0) = y_0 \in Z_0\,. \tag{9}$$

G. Duvant and J.-L. Lions (1976), W. Han and M. Sofonea (2000), K.L. Kuttler and M. Shillor (1999)

Definition 1.1 *A function $z(\cdot) \in \mathcal{W}_T \cap C(0,T;Z_0)$ is said to be a solution of (8), (9) on $(0,T)$ if there exists a function $u(\cdot) \in L^2(0,T;U)$ such that for a.a. $t \in (0,T)$ the inequality (8), (9) is satisfied and $\int_0^T \psi(z(t))dt < +\infty$. The pair $\{z(\cdot), u(\cdot)\}$ is called a* response *of (8), (9); $u(\cdot)$ is an associated* selection*.*

Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two quadratic forms on $Z_1 \times U$. The *class $\mathcal{N}(\mathcal{F},\mathcal{G})$ of nonlinearities* for (8), (9) consists of all maps (6) such that the following condition is satisfied:

For any $T > 0$ and any two functions $z(\cdot) \in L^2(0,T;Z_1)$ and $u(\cdot) \in L^2(0,T;U)$ with

$$u(t) \in \varphi(t, Cz(t)) \quad \text{for a.a.} \quad t \in [0,T]\,, \tag{10}$$

it follows that

$$\mathcal{F}(z(t), u(t)) \geq 0 \quad \text{for a.a.} \quad t \in [0,T]\,, \tag{11}$$

and there exists a continuous function $\Phi : Z_1 \to \mathbb{R}_+$ s. t.

$$\int_s^t \mathcal{G}(z(\tau), u(\tau))d\tau \geq \Phi(z(t)) - \Phi(z(s)) \geq -\Phi(z(s)) \tag{12}$$
$$\text{for all } 0 \leq s < t \leq T.$$

2. Frequency-domain conditions for finite-time stability

Suppose that $0 \le t_0 < T_0$, $T > 0$ with $t_0 + T < T_0$, and $0 < \alpha < \beta$ are arbitrary numbers.

Definition 2.1 *The inequality (8), (9), Sect. 1, is called* (α, β, t_0, T)-stable *if for any solution* $z(\cdot)$ *from* $\|z(t_0)\|_0 < \alpha$ *it follows that* $\|z(t)\|_0 < \beta$ *for all* $t \in [t_0, t_0 + T)$.

Theorem 2.1 *The inequality (8), (9), Sect. 1, is* (α, β, t_0, T)*-stable if there exist a continuous functional* V *on* $Z_0 \times [t_0, t_0 + T)$ *and an integrable on* $J := [t_0, t_0 + T)$ *real-valued function* g *such that the following conditions are satisfied:*

$$(i) \qquad V(z(t), t) - V(z(s), s) < \int_s^t g(\tau) d\tau \tag{1}$$

for all $s, t \in J, s < t$*, and all functions* $z(\cdot) \in \mathcal{W}(J, Z_0) \cap C(J, Z_0)$ *such that*

$$\alpha \le \|z(t)\|_0 \le \beta \text{ for all } t \in J;$$

$$(ii) \qquad \int_s^t g(\tau) d\tau \le \min_{z \in Z_0 : \|z\|_0 = \beta} V(z, t) - \max_{z \in Z_0 : \|z\|_0 = \alpha} V(z, s) \tag{2}$$

for all $s, t \in J, s < t$.

ODE-case: L. Weiss and E.F. Infante (1965), A.N. Michel and D.W. Porter (1972)

The pair of operators (A, B) from (8), (9), Sect. 1, is *stabilizable*, if there exists an operator $S \in \mathcal{L}(Z_0, U)$ such that any solution $z(\cdot)$ of the Cauchy-problem $\dot{z} = (A + BS)z, z(0) = z_0$, decreases exponentially as $t \to +\infty$, i.e.,

$$\exists\, c > 0 \;\exists\, \varepsilon > 0 : \|z(t)\|_0 \le c\, e^{-\varepsilon t} \|z_0\|_0 , \quad \forall\, t \ge 0 .$$

The complexification of a real Hilbert space $(H, (\cdot,\cdot)_H)$ and a real linear operator L are denoted by $(H^c, (\cdot,\cdot)_{H^c})$ and L^c, respectively. For a real quadratic form $\mathcal{F}$ we denote their Hermitian extension by $\mathcal{F}^c$. We introduce for the pair (A, B) from (8), (9), Sect. 1, the frequency-domain characteristic

$$\chi(i\omega) = (i\omega I^c - A^c)^{-1} B^c \; . \tag{3}$$

Let us also introduce the real Hilbert space Y with scalar product $(\cdot,\cdot)_Y$ and norm $\|\cdot\|_Y$.

For this we define the linear operators $M \in \mathcal{L}(Z_1, Y)$ and $N \in \mathcal{L}(U, Y)$ s.t. for any response $\{z(\cdot), u(\cdot)\}$ of (8), (9), Sect. 1,

$$y(t) := Mz(t) + Nu(t) \in Y \; , \; t \in [0, T_0] \; . \tag{4}$$

Lemma 2.1 *Suppose that the pair (A, B) is stabilizable, $\varphi \in \mathcal{N}(\mathcal{F}, \mathcal{G})$, and there exist numbers $\varepsilon > 0$ and $\delta \in \mathbb{R}$ such that*

$$\mathcal{F}^c(\chi(i\omega)u, u) \; + \; \mathcal{G}^c(\chi(i\omega)u, u) \; + \; \delta\|\big[M^c\chi(i\omega) + N^c\big]u\|^2_{Y^c} \\ + \; \varepsilon\big[\|\chi(i\omega)u\|^2_{Z_1^c} \; + \; \|u\|^2_{U^c}\big] \leq 0 \; , \tag{5}$$

for all $\omega \in \mathbb{R}$ with $i\omega \notin \sigma(A^c)$ and all $u \in U^c$. Then there exists a real operator
$P = P^* \in \mathcal{L}(Z_0, Z_0)$ such that for any response $\{z(\cdot), u(\cdot)\} \not\equiv \{0, 0\}$ of (8), (9), Sect. 1, and for any s, t with $0 < s \leq t < T_0$ we have

$$(z(t)\,,\; Pz(t))_0 - (z(s)\,,\; Pz(s))_0 < \\ \int_s^t [\psi(-Pz(\tau) + z(\tau)) - \psi(z(\tau)) + \\ \delta\|Mz(\tau) + Nu(\tau)\|^2_Y]d\tau + \Phi(z(s)) \; . \tag{6}$$

A.L. Likhtarnikov and V.A. Yakubovich (1976)

V.A. Brusin (1976)

L. Pandolfi (1998)

Corollary 2.1 *Suppose that there exist operators* $P = P^* \in \mathcal{L}(Z_0, Z_0)$, $M \in \mathcal{L}(Z_1, Y)$, $N \in \mathcal{L}(U, Y)$ *and a number* $\delta \in \mathbb{R}$ *such that the inequality (6) is satisfied. Assume that* $0 < \alpha < \beta$ *and* $T > 0$ *with* $t_0 + T < T_0$ *are arbitrary numbers. Then for any response* $\{z(\cdot), u(\cdot)\}$ *of (8), (9), Sect. 1, with* $\alpha \leq \|z(t)\|_0 \leq \beta$ *for* $t \in [t_0, t_0 + T)$ *and any* s, t *with* $0 < s \leq t \leq t_0 + T$ *the inequality*

$$V(z(t)) - V(z(s)) < \int_s^t g(\tau) d\tau \tag{7}$$

is satisfied.

Here

$$V(z) := (z, Pz)_0 \ , \ z \in Z_0, \quad \text{and} \tag{8}$$

$$\begin{aligned} g(t) := \sup[&\psi(-Pz(t) + z(t)) - \psi(z(t)) + \\ &\delta \|Mz(t) + Nu(t)\|_Y^2 + \frac{1}{T} \Phi\left(z(0)\right)] \ , \end{aligned} \tag{9}$$

where the supremum is taken over all pairs $\{z(\cdot), u(\cdot)\}$ *with* $z(\cdot) \in L^2(t_0, t_0 + T; Z_0)$, $u(\cdot) \in L^2(t_0, t_0 + T; U)$ *such that* $\alpha \leq \|z(t)\|_0 \leq \beta$, $\mathcal{F}(z(t), u(t) \geq 0$ *and* $\mathcal{G}(z(t), u(t)) \geq -\Phi(z(t))$ *for a.a.* $t \in [t_0, t_0 + T)$.

Theorem 2.2 *Suppose that the assumptions of Lemma 3.1 are satisfied and, with the operators* P, M, N *and the number* δ *from this lemma, the inequality (6) holds. If with the functions* V *and* g *from (8), (9) and arbitrary* s, t *with* $t_0 \leq s \leq t \leq t_0 + T$ *the inequality*

$$\int_s^t g(\tau) d\tau \leq \min_{z \in Z_0 : \|z\|_0 = \beta} V(z) - \max_{z \in Z_0 : \|z\|_0 = \alpha} V(z) \tag{10}$$

is satisfied, then the variational inequality (8), (9), Sect. 1, is (α, β, t_0, T)*-stable.*

3. Determining observation functionals via the algebra of operator-symbols

Recall that $H^\infty(G)$ is a subspace of $L^2(\mathbb{R}^m)$ consisting of all such (complex valued) functions the Fourier transform of which is compactly supported in G.

Yu.A. Dubinskii (1982)

Consider the Cauchy problem for the variational equation

$$\frac{\partial z}{\partial t} = A(D)z + B(D)\, u(x,t)\ , \quad z(x,0) = z_0(x)\ , \tag{1}$$

$$w(x,t) = C(D)z(x,t), \quad u(x,t) \in \varphi(t, w(x,t)), \tag{2}$$

where $A(D), B(D)$ and $C(D)$ are ΨDO's with constant coefficients the symbols of which are real analytic in G. The action of these linear operators is supposed as

$$A(D) : [H^\infty(G)]^n \to [H^{-\infty}(G)]^n\ , \tag{3}$$

$$B(D) : [H^\infty(G)]^k \to [H^{-\infty}(G)]^n\ , \tag{4}$$

$$C(D) : [H^\infty(G)]^n \to [H^\infty(G)]^l\ , \tag{5}$$

where n, k and l are natural numbers.

We assume the representation

$$A(D) = \sum_{|\alpha|=0}^{\infty} A_\alpha D^\alpha\ , \quad B(D) = \sum_{|\alpha|=0}^{\infty} B_\alpha D^\alpha\ ,$$

$$C(D) = \sum_{|\alpha|=0}^{\infty} C_\alpha D^\alpha \tag{6}$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ are multiindices, $D = (D_1, \dots, D_m)$ with $D_j = \frac{1}{i}\frac{\partial}{\partial x_j} (j = 1, \dots, m)$ are again the elementary differential operators and A_α, B_α and C_α are constant $n \times n$-, $n \times k$- and $l \times n$-matrices, respectively. The associated symbols

$$A(\xi) = \sum_{|\alpha|=0}^{\infty} A_\alpha \xi^\alpha \ , \quad B(\xi) = \sum_{|\alpha|=0}^{\infty} B_\alpha \xi^\alpha \ , \quad C(\xi) = \sum_{|\alpha|=0}^{\infty} C_\alpha \xi^\alpha \tag{7}$$

are real analytic in G matrix-valued functions. The nonlinear part of (1), (2) is given as set-valued map

$$\varphi : \mathbb{R}_+ \times [H^\infty(G)]^l \to 2^{[H^\infty(G)]^k} \ . \tag{8}$$

The initial function is assumed as $z_0 \in [H^\infty(G)]^n$. The equation (1), (2), (8) is understood in the sense of distributions.

The class of nonlinearities (8) is described by an Hermitian form $\mathcal{F}$ on $[H^\infty(G)]^n \times [H^\infty(G)]^k$ given by

$$\mathcal{F}(z, u) = (F_1(D)z, z)_{-\infty,\infty} + 2\,\mathrm{Re}\,(F_2(D)u, z)_{-\infty,\infty} + (F_3(D)u, u) \tag{9}$$

where

$$\left.\begin{array}{rcl} F_1(D) & = & F_1^*(D) \in \mathcal{L}([H^\infty(G)]^n \,, \, [H^{-\infty}(G)]^n) \ , \\ & & F_2(D) \in \mathcal{L}([H^\infty(G)]^k \,, \, [H^{-\infty}(G)]^n) \ , \\ F_3(D) & = & F_3^*(D) \in \mathcal{L}([H^\infty(G)]^k \,, \, [H^\infty(G)]^k) \end{array}\right\} \tag{10}$$

are ΨDO's with constant coefficients, the symbols

$$F_i(\xi) = \sum_{|\alpha|=0}^{\infty} F_{i\alpha} \xi^\alpha \ , i = 1, 2, 3,$$

of which are real analytic in G.

M. Taylor (1981), F. Treves (1980)

The finite-time stability problem can be reduced to the problem of solving the Lur'e operator equations

$$\left.\begin{array}{lcl} A^*(D)P(D) + P(D)A(D) + L(D)L^*(D) & = & -F_1(D)\ , \\ P(D)B(D) - L(D)K(D) & = & -F_2(D)\ , \\ K(D)K^*(D) & = & -F_3(D)\ , \end{array}\right\} \tag{11}$$

where $P(D) = P^*(D) \in \mathcal{L}\left([H^{\pm\infty}(G)]^n\, ,\ [H^{\pm\infty}\ (G)]^n\right)$,
$L(D) \in \mathcal{L}([H^{\infty}(G)]^k,\ [H^{-\infty}(G)]^n)$, and
$K(D) = K^*(D) \in \mathcal{L}\left([H^{\infty}(G)]^k, [H^{\infty}(G)]^k\right)$
are unknown ΨDO's with constant coefficients the symbols of which are real analytic in G.

In order to solve (11) we use, as it was done in, the isomorphism between the algebra of ΨDO's with constant coefficients and analytic symbols and the algebra of analytic matrix-valued functions which describe the symbols. This means that we have to solve the matrix-valued Lur'e problem

$$\left.\begin{array}{lcl} A^*(\xi)P(\xi) + P(\xi)A(\xi) + L(\xi)L^*(\xi) & = & -F_1(\xi)\ , \\ P(\xi)B(\xi) - L(\xi)K(\xi) & = & -F_2(\xi)\ , \\ K(\xi)K^*(\xi) & = & -F_3(\xi)\ , \end{array}\right\} \tag{12}$$

where the given real analytic in G matrix-valued functions $A(\xi), B(\xi), F_1(\xi), F_2(\xi)$ and $F_3(\xi)$ are of order $n \times n, n \times k$, $n \times n, n \times k$ and $k \times k$, respectively, and the unknown real analytic in G matrix-valued functions $P(\xi) = P^*(\xi), L(\xi)$ and $K(\xi) = K^*(\xi)$ are of order $n \times n, n \times k$ and $k \times k$, respectively.

A.L. Likhtarnikov (1989)

Remark 3.1 Suppose that the ΨDO's A, B, C, F_1, F_2 and F_3 in (12) depend continuously on the time $t \in \mathbb{R}$. Then the calculation of a time-dependent quadratic measurement functional V in the sense of Theorem 1.1, which is represented by a pseudo differential operator $P(t,D)$ in $H^\infty(G)$ can be reduced to the problem of solving *differential Lur'e-Riccati equations* for absolute continuous in t and real analytic in ξ symbols $P(t,\xi), L(t,\xi)$ and K (t,ξ) in $\mathbb{R}\times G$:

$$\left.\begin{aligned} \dot P(t,\xi) + A^*(t,\xi)P(t,\xi) + P(t,\xi)A(t,\xi) + L(t,\xi)L^*(t,\xi) & \\ = -F_1(t,\xi)\ , & \\ P(t,\xi)\,B(t,\xi) - L(t,\xi)\,K(t,\xi) = -F_2(t,\xi)\ , & \\ K(t,\xi)\,K^*(t,\xi) = -F_3(t,\xi)\ . & \end{aligned}\right\} \tag{13}$$

Sufficient frequency-domain conditions for the solvability of (13) are given in V.A. Yakubovich, 1986 (periodic in t case) and in A.V. Savkin, 1993 (general bounded in t case).

Example 3.1 Consider the heat equation on an infinite bar

$$\tfrac{\partial z}{\partial t} = a\,\tfrac{\partial^2 z}{\partial x^2} + u(x,t)\ , \tag{14}$$

$$u(x,t) = \varphi\,(z(x,t))\ , \quad z(x,0) = z_0(x)\ , \tag{15}$$

where $a > 0$ is a parameter and $\varphi : \mathbb{R} \to \mathbb{R}$ is a given nonlinear function which satisfies the inequality

$$\|\varphi\,(z)\| \le \mu\|z\| \quad , \quad \forall z \in L^2(\mathbb{R})\ . \tag{16}$$

Here $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R})$ and $0 < \mu < 2$ is also a parameter. It follows that φ is characterized by the quadratic form in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ given as

$$\mathcal{F}(z,u) := \mu^2\|z\|^2 - \|u\|^2\,. \tag{17}$$

$$A(\xi) = -a\,\xi^2, \quad B(\xi) \equiv 1\ , \tag{18}$$

$$F_1(\xi) \equiv \mu^2\,, \quad F_2(\xi) \equiv 0\ , \quad F_3(\xi) \equiv -1\,. \tag{19}$$

$$\chi(i\omega,\xi) = (i\omega + a\,\xi^2)^{-1}\ . \tag{20}$$

$$G_1^{'} := \left\{(\xi,\omega)\in\mathbb{R}^2 : (1 - a\,\xi^2)^2 + \omega^2 < 1\right\}\ . \tag{21}$$

$$\mathcal{F}(\chi(i\omega,\xi)u, u, \xi) =: u^*\Pi(i\omega,\xi)u\ .$$

$$\Pi(i\omega,\xi) = \mu^2|\chi(i\omega,\xi)|^2 - 1\ . \tag{22}$$

$$\begin{aligned} G^{'} &:= \left\{(\omega,\xi)\in\mathbb{R}^2 : \xi > 0,\ \mu^2 < \omega^2 + a^2\xi^4 < 2\,a\,\xi^2\right\}\ . \\ G := \text{proj}\ G^{'} &= \left\{\xi > 0\ :\ \mu^2 < a^2\xi^4 < 2\,a\,\xi^2\right\} \\ &= \left\{\xi\in\mathbb{R} : \sqrt{\mu/a} < \xi < \sqrt{2/a}\right\}\ . \end{aligned}$$

The pair $(A(\xi), B(\xi))$ from (18) is stabilizable in G.

Lur'e equations in G:

$$-2\,P(\xi)\,a\,\xi^2 + L^2(\xi) = -\mu^2\ , \tag{23}$$

$$P(\xi) - L(\xi)\,K(\xi) = 0\ , \tag{24}$$

$$K^2(\xi) = 1\ . \tag{25}$$

$\Longrightarrow K(\xi) \equiv 1$, $P(\xi) \equiv L(\xi)$.

Equation for $P(\cdot)$ in G:

$$P^2(\xi) - 2\,a\,\xi^2 P(\xi) + \mu^2 = 0\ , \tag{26}$$

$$P(\xi) = a\,\xi^2 + \sqrt{a^2\xi^4 - \mu^2}\ .$$