Bohr almost periodic in time temperature fields in a body with nonlinear boundary heating

V. Reitmann*)

Max Planck Institute for the Physics of Complex Systems

Dresden, Germany

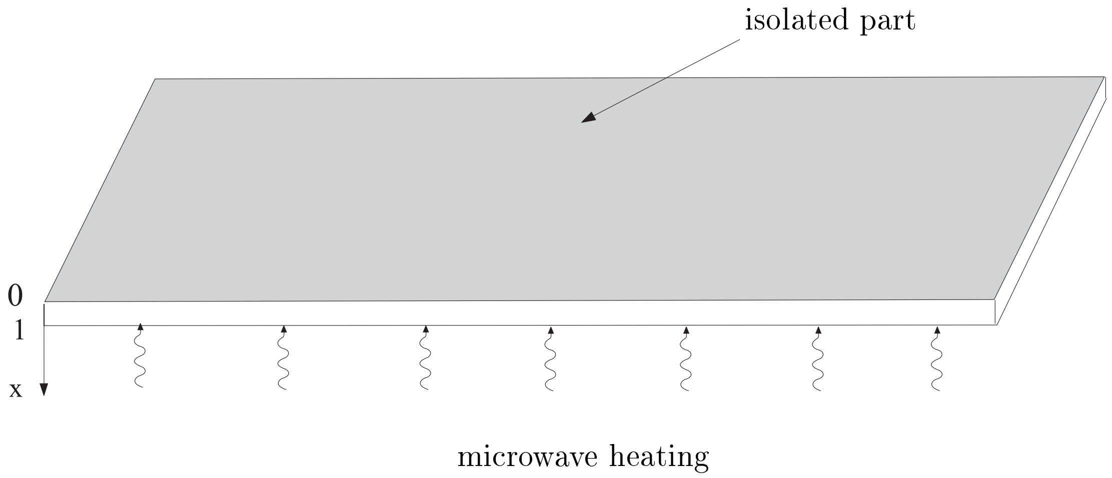
January 26, 2006

Abstract We consider a control problem for the heating process of a finite solid body. The heat flux within the body is modelled by the heat equation with nonlinear Neumann boundary conditions according to Newton's law. As input at a part of the boundary we take the nonlinearly transformed and modulated heat production of a separate heater which is given by a nonlinear Duffing-type ODE. This ODE depends on measurements of the temperature within the body and on Bohr resp. Stepanov almost periodic in time forcing terms. The physical problem is generalized to a bifurcation problem for non-autonomous evolution systems in Hilbert spaces. Using energy-type functionals, invariant cones and monotonicity properties of the nonlinearities in certain Sobolev spaces, we show the existence and uniqueness of an asymptotically stable and almost periodic in time temperature field which is more regular than other solutions and localized in space ("breather"). We also derive sufficient conditions to avoid a "temperature collapse" in the coupled system. A possible application of our localization result seems to be the control of the temperature distribution within a body in cancer therapy ("regional hyperthermia").

*)Supported by the DFG-Schwerpunktprogramm 1114: Mathematical methods for time series analysis and digital image processing

1 The physical problem

Suppose that there is a thin elastic plate which is heated from below:



The elastic equation for the *displacements* $u(x,t)$ is

$$u_{tt} - \sigma_x = 0$$

where the *stress* σ is defined by the *Duhamel-Neumann law*

$$\sigma = \gamma_1 u_x - \gamma_2 \theta \ , \quad \theta = \theta(x,t) \qquad \text{the } \textit{temperature.}$$

This gives the *thermo-elastic equation*

$$u_{tt} = \gamma_1 u_{xx} - \gamma_2 \theta_x \tag{1}$$

with the temperature gradient θ_x and mixed boundary conditions

$$\gamma_3 u_{x_{|x=0}} + \gamma_4 \, \theta_{|x=0} = 0 \ ,$$
$$\gamma_5 u_{x_{|x=0}} + \gamma_6 \, \theta_{|x=1} = 0$$

and initial conditions $u(\cdot,0) = u_0$, $u_t(\cdot,0) = u_1$. The heat equation for the plate is

$$\theta_t = \delta_1 \theta_{xx} - \delta_2 \, \theta + \delta_6 u_{xt} \tag{2}$$

where $\delta_6 u_{xt}$ is the *elastic compression term*, with boundary conditions of Neumann-type:

$$\theta_{x_{|x=0}} = 0 \ , \quad \theta_{x_{|x=1}} = \theta\,(1,t) - \theta_{ext}(t) \qquad \text{(Newton's law)}$$
or
$$\theta_{x_{|x=0}} = 0 \ , \quad \theta_{x_{|x=1}} = \theta^4(1,t) - \theta^4_{ext}(t) \qquad \text{(Stefan Boltzmann law)} \, .$$

Here θ_{ext} is the heat supply arising from external heat source.
We assume *Maxwell's equation* for heat generation by induction (for convection heating we have other boundary conditions) $\Rightarrow$ Energy equation for the *heat power* $w(t)$

$$\dot{w}(t) = \delta_7 w(t) + \alpha(t) \, . \tag{3}$$

The *control* $\alpha(\cdot)$ is assumed as

$$\alpha(t) = \underbrace{\int_0^1 \theta(x,t)k(x)dx}_{\text{measurement}} \qquad + \qquad \delta_8 \left[\underbrace{\phi\,(t,w)}_{\substack{\text{nonlinear}\\ \text{function}}} + \underbrace{g(t)}_{\substack{\text{external}\\ \text{excitation}}} \right].$$

(Point measurement $\alpha(t) = \theta(x_0, t),\ x_0$ fixed, is possible.)

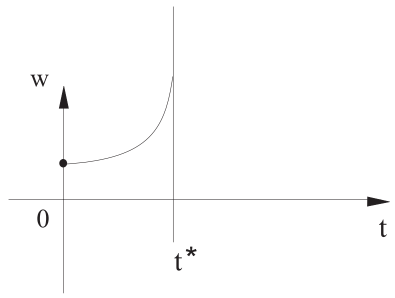
In system (1) – (3) various types of instabilities or bifurcations may occur:

Example 1.1

$$\dot{w} = w^4$$

$$\text{solution :} \quad w(t) = \left(\frac{w(0)^3}{1 - 3\,w(0)^3 t} \right)^{1/3}$$

$$\Rightarrow \text{ blow up at } t^* = \frac{1}{3w(0)^3}$$



hot spot

□

physical	mathematical
a) hot spots	blow up of solutions
b) elastic buckling	loss of dissipativity
c) cracks	loss of regularity of the solution
d) forced thermo-elastic vibrations	existence of stable almost-periodic solutions

Standard theory for the proof of forced period solutions (Schauder's fixed point theorem, Bogolyubov/Krylov method) is *not applicable* since:

1) all points of the spectrum of the associated linearization problem are nonisolated.
2) the operators on the space of a.p. functions are *not completely* continuous, i.e. do not map bounded sets in compact sets.

$\Rightarrow$ need for monotonicity methods

For simplicity we consider the restricted *boundary control problem* for the temperature

$$\theta_t = \delta_1 \theta_{xx} - \delta_2\,\theta\,, \qquad \delta_1 > 0, \delta_2 > 0 \tag{4}$$

$$\theta_{x_{|x=0}} = 0\,, \quad \theta_{x_{|x=1}} = \delta_3[\phi(t,w) + g(t)]\,, \qquad \delta_3 \in \mathbb{R}$$

$$\dot{w} = \int_0^1 \theta(x,t)\,k(x)\,dx + \delta_4[\phi(t,w) + g(t)]\,, \qquad k(\cdot) \text{ kernel function,} \quad \delta_4 < 0 \tag{5}$$

$$\phi(t,w) = w - \delta_5(t) w^3 \qquad \textit{Duffing-type} \text{ nonlinearity,} \qquad \delta_5(t) \geq 0 \text{ a.e.}$$

Remark 1.1 (4), (5) is also a 1-dimensional dynamic model for a *nuclear reactor*:

θ reactor temperature

w reactor power

$$
\begin{cases}
\theta_t = \delta_1 \theta_{xx} - \delta_2 \theta + \delta_3 [\phi(t,w) + g(t)] & (4') \\
\alpha_1 \theta_{x_{|x=0}} + \alpha_2 \, \theta(0,t) = 0 & \\
\alpha_3 \theta_{x_{|x=1}} + \alpha_3 \, \theta(1,t) = 0 & \\
\dot{w} = \int_0^1 \theta(x,t) \, k(x) \, dx & (5')
\end{cases}
$$

(4'), (5') is not with boundary heating $\Rightarrow$ control operator is bounded $\square$

2 Almost-periodic functions

Let Trig $(\mathbb{R}, \mathbb{C})$ be the space of all trigonometric polynomials with coefficients in $\mathbb{C}$, i.e.

$$
\mathrm{Trig}(\mathbb{R};\mathbb{C}) := \big\{ T : \mathbb{R} \to \mathbb{C} \,|\, T(x) = \sum_{k=1}^{n} a_k e^{i\lambda_k x} \,, \qquad n \in \mathbb{N}, a_k, \lambda_k \in \mathbb{C} \big\} \;.
$$

In Trig $(\mathbb{R};\mathbb{C})$ are various metrics possible:

$$
d_B(f,g) : = \sup_{t \in \mathbb{R}} |f(x) - g(x)| \qquad \textit{Bohr distance}
$$

$$
d_{S^2}(f,g) : = \sup_{x \in \mathbb{R}} \left(\int_x^{x+1} |f(t) - g(t)|^2 \right)^{1/2} dt \qquad \textit{Stepanov distance (of order 2)}
$$

Definition 2.1 *A function* $f : \mathbb{R} \to \mathbb{C}$ *is called* Bohr a.p. (Stepanov a.p.) *if there is a sequence* $\{f_n\} \subset$ *Trig* $(\mathbb{R};\mathbb{C})$ *s.th.* $d_B(f_n, f) \to 0$ $(d_{S^2}(f_n, f) \to 0)$ *as* $n \to \infty$.

Notation: CAP $(\mathbb{R};\mathbb{C})$ continuous Bohr a.p. functions
$S^2(\mathbb{R};\mathbb{C})$ Stepanov a.p. functions

Fact: If $f \in S^2(\mathbb{R};\mathbb{C})$ is equicontinuous $\Rightarrow f \in CAP(\mathbb{R};\mathbb{C})$.

Definition 2.2 *A subset* $S \subset \mathbb{R}$ *is* relatively dense *if there is a compact interval* $K \subset \mathbb{R}$ *such that* $(s + K) \cap S \neq \emptyset, \forall s \in \mathbb{R}$. *A bounded continuous function* $f : \mathbb{R} \to \mathbb{C}$ *is said to be Bohr a.p. if for any* $\varepsilon > 0$ *the set*

$$
\{\tau \in \mathbb{R} | \sup_{s \in \mathbb{R}} | f(s + \tau) - f(s) | \leq \varepsilon\}
$$

of ε-almost periods *is relatively dense in* $\mathbb{R}$.

(A1)
- $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous; $\phi\,(t,0) = 0, \quad \forall t \in \mathbb{R}$;
- The family $\{\phi(\cdot, w) \,|\, w \in M, M \subset \mathbb{R}$ bounded $\}$ is uniformly Bohr a.p. for any bounded set $M \subset \mathbb{R}$;
- $g : \mathbb{R} \to \mathbb{R}$, $g \in L^2_{loc}(\mathbb{R};\mathbb{R}) \cap S^2(\mathbb{R};\mathbb{R})$
- $\exists\, \kappa_1 > 0, 0 \leq \kappa_2 < \kappa_3 < \infty$, $q_1 < q_2, r_2 < r_1$ s.th. :

  a) $q_1 < g(t) < q_2$ for a.e. t from any compact time interval;

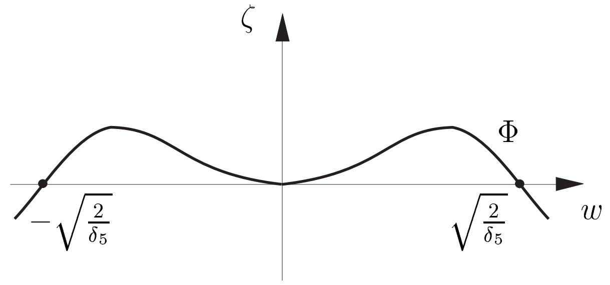
  b) $(\phi(t,w) + q_i)(w - r_i) \leq \kappa_1 (w - r_i)^2, \; i = 1,2 \qquad \forall\, t \in \mathbb{R}, \forall w \in [r_2, r_1]$;

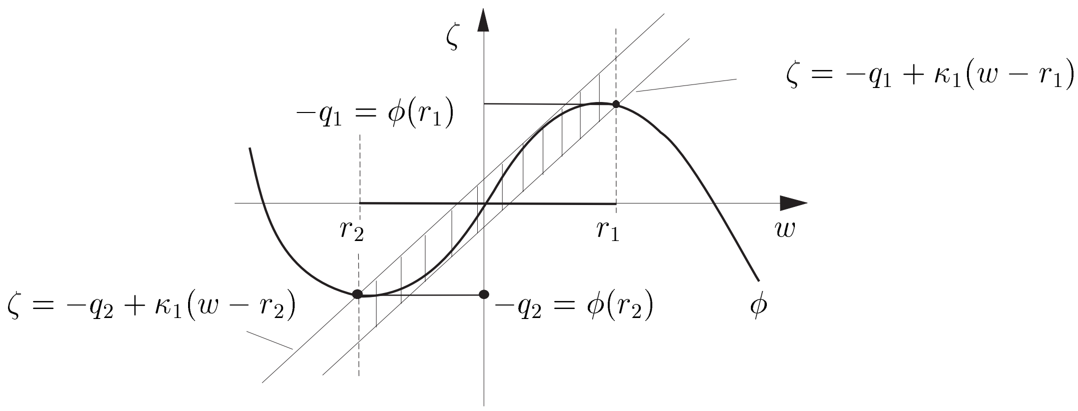
  c) $\kappa_2 (w_1 - w_2)^2 \leq (\phi(t,w_1) - \phi(t,w_2))(w_1 - w_2) \leq \kappa_3 (w_1 - w_2)^2,$
  $\forall\, t \in \mathbb{R}, \quad \forall\, w_1, w_2 \in [r_2, r_1]$.

Nonlinearity and forcing function:

$$\phi(w) = w - \delta_5 w^3\ ,\ \delta_5 > 0$$

$$\phi = \Phi'\ , \qquad \Phi(w) = \frac{w^2}{2} - \frac{\delta_5}{4} w^4 \qquad \textit{double-well potential}$$





$$r_2 = -\frac{1}{\sqrt{3\,\delta_5}} + \varepsilon$$

$$r_1 = \frac{1}{\sqrt{3\,\delta_5}} - \varepsilon\ , \qquad \varepsilon > 0 \qquad \text{small}$$

$$q_2 = -\phi(r_2)\ ,\ q_1 = -\phi(r_1)$$

$$\kappa_1 = \frac{\delta_2^2}{4}$$

3 Abstract formulation

Write (4), (5) as ODE in Hilbert space

$$\dot v = A_0 v + B_0[\phi(t,w) + g(t)] \tag{6}$$

$$\dot w = C_0 v + d_0[\phi(t,w) + g(t)]\ , \tag{7}$$

$V_1 := W^{1,2}(0,1)$,	$V_0 := L_2(0,1)$,	$V_{-1} = V_1^*$
space of test functions	state space	dual space (w.r.t. V_0)

$$(v,\vartheta)_1 := \int_0^1 [v\vartheta + v_x\vartheta_x]\,dx\ , \qquad v,\vartheta \in V_1$$

$A_0 : V_1 \to V_{-1}$ is given by

$$(A_0 v, \vartheta) = -\int_0^1 [\delta_1 v'(x)\vartheta'(x) + \delta_2 v(x)\vartheta(x)]\, dx$$

$B_0 : \mathbb{R} \to V_{-1}$ *(Control operator)*

$$(B_0\, \xi, v) = \delta_1 \xi v(1)\ , \quad \forall \xi \in \mathbb{R}\ , \quad \forall v \in V_1$$

i.e. $B_0 = \delta_1 \delta(x-1)$ Dirac's δ-function concentrated at $x = 1$
$C_0 : V_0 \to \mathbb{R}$ *(measurement operator)* is given by

$$C_0 v := \int_0^1 k(x) v(x) dx\ , \quad \forall v \in V_0.$$

(A2) For any $T > 0$ and $(f_1, f_2) \in L^2(0, T; V_{-1} \times \mathbb{R})$ the problem

$$\begin{aligned} \dot v &= A_0 v + f_1\ , \\ \dot w &= C_0 v + f_2 \end{aligned} \tag{8}$$

is *well-posed*, i.e. there exists a solution depending continuously on the initial data and $|f_1|, |f_2|$.

(A3) $\exists \lambda > 0$ such that $A_0 + \lambda I$ is a *stable operator*, i.e. any solution of $\dot v = (A_0 + \lambda I) v, v(0) = v_0$, tends to zero as $t \to +\infty$.

(A4) The solution of (8) and of the adjoint system

$$\begin{aligned} \dot\psi &= -(A_0^* - \lambda I)\psi + f_1 \\ \dot z &= -C^* \psi - \lambda z + f_2 \end{aligned}$$

are continuous in t in the norm of $V_1 \times \mathbb{R}$.

(A5) The pair (A_0, B_0) is L^2-controllable, i.e. for any $v_0 \in V_0$ there exists a control $\alpha(\cdot) \in L^2(0, \infty; \mathbb{R})$ such that $\dot v = A_0 v + B_0 \alpha, v(0) = v_0$ has solutions on $\mathbb{R}_+$.

(A6) Let $\chi(p) = C_0(A_0 - pI)^{-1} B_0, p \in \mathbb{C}$, the transfer operator function of (6), (7). There exists a $\lambda > 0$ such that

$$\lambda d_0 + \mathrm{Re}(-i\omega - \lambda)\chi(i\omega - \lambda) + \kappa_1 \mid \chi(i\omega - \lambda) - d_0 \mid^2 \le 0, \omega \in \mathbb{R}_+\ .$$

(Frequency domain condition for the existence of invariant cones)

(A7) There exists a variational solution of (6), (7).

(A8)

$$\frac{1}{\kappa_3 - \kappa_2} + \mathrm{Re}\, \frac{\chi(i\omega) - d_0}{i\omega + \kappa_2(\chi(i\omega) - d_0)} > 0\ , \quad \forall \omega \in \mathbb{R}.$$

(Frequency domain condition for local dissipativity)

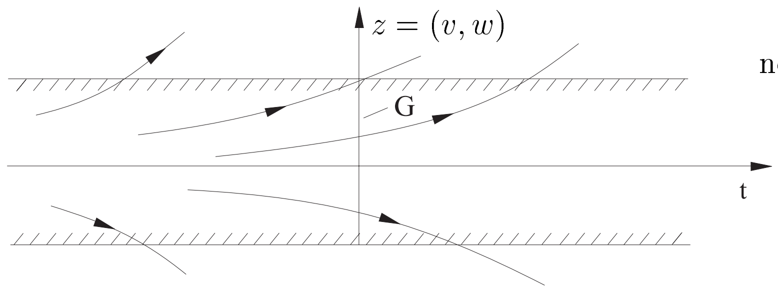
(A9) The operator $\begin{bmatrix} A_0 & \kappa_2 B_0 \\ C_0 & \kappa_2 d_0 \end{bmatrix}$ is stable.

Theorem 3.1 *Assume that for (6), (7) the assumptions* **(A1)** – **(A9)** *are satisfied. Then we have:*
a) There exists a closed, positively invariant and convex set G such that

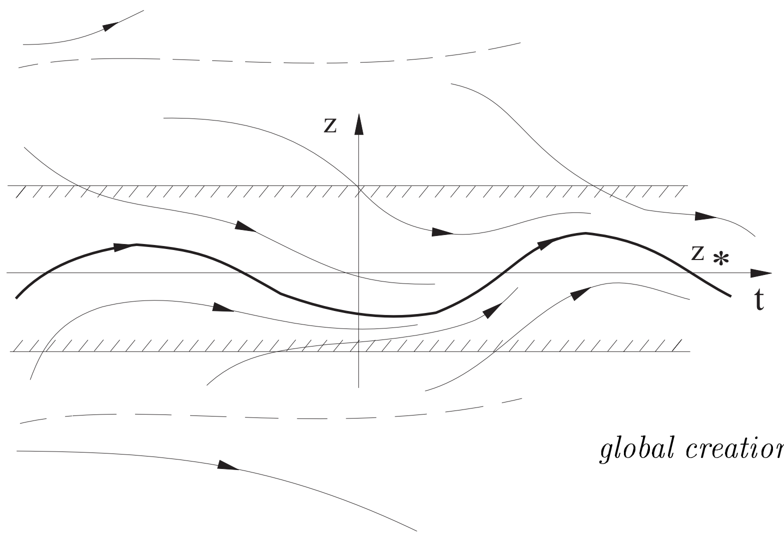
$$\{(v,w) \in V_1 \times \mathbb{R} \,|\, v = 0, w \in [r_2, r_1]\} \subset G \subset \{(v,w) \in V_1 \times \mathbb{R} \,|\, w \in [r_2, r_1]\}.$$

b) For any $g \in S^2(\mathbb{R};\mathbb{R})$ system (6), (7) has a unique Bohr a.p. solution (v_*, w_*) in G and this solution is exponentially stable inside G.

- *Geometrical interpretation*



$\varepsilon < \varepsilon_1$
no dissipativity



$\varepsilon_1 < \varepsilon < \varepsilon_2$
global creation of an a.p. solution

4 Application to the heating problem / results

(A7): *Variational solution of (6), (7)*

A pair of functions $(\theta(x,t), w(t))$ is a *weak solution* of (4), (5) on $(0,T)$ if

$$\theta(\cdot,t) \in W^{1,2}(0,1)\,, \qquad w, \dot{w} \in L^2(0,T)\,,$$

$$\int_0^T \left\{ \int_0^1 [\theta\eta_t - (\delta_1\theta_x\eta_x + \delta_2\theta\eta)]\,dx + \delta_1\delta_3[\phi(t,w) + g(t)]\,\eta\,(1,t) \right\} dt = 0\,, \tag{9}$$

$$\int_0^T \left\{ w(t)\zeta(t) + \left(\int_0^1 \theta(x,t)k(x)\,dx + \delta_4[\phi(t,w) + g(t)] \right) \zeta\,(t) \right\} dt = 0\,, \tag{10}$$

$\forall$ smooth test function $\eta(x,t)$, $\eta(x,0) = \eta(x,1) = 0$,

$\forall$ smooth test function $\zeta(t)$, $\zeta(0) = \zeta(T) = 0$.

(A6):

Transfer function: $\chi(p) = \int_0^1 \tilde{\theta}(x,p)\,dx$ where $\tilde{\theta}(x,p)$ is the solution of the BVP $(k(x) \equiv 1,\ \delta_3 = 1,\ \delta_4 = -1,\ \delta_5(t) \equiv \delta_5)$:

$$p\,\tilde{\theta} = \delta_1\tilde{\theta}'' - \delta_2\tilde{\theta}\,,$$

$$\tilde{\theta}'_{|x=0} = 0, \quad \tilde{\theta}'_{|x=1} = 1\,,$$

$$\Rightarrow \quad \tilde{\theta}(x,p) = \frac{\cosh\sqrt{p+\delta_2}\,x}{\sqrt{p+\delta_2}\sinh\sqrt{p+\delta_2}}\,,$$

$$\Rightarrow \quad \chi(p) = \frac{1}{\sqrt{p+\delta_2}\sinh\sqrt{p+\delta_2}} \int_0^1 \cosh\sqrt{p+\delta_2}\,dx = \frac{1}{p+\delta_2}\,,$$

$\Rightarrow$ sufficient to assume that

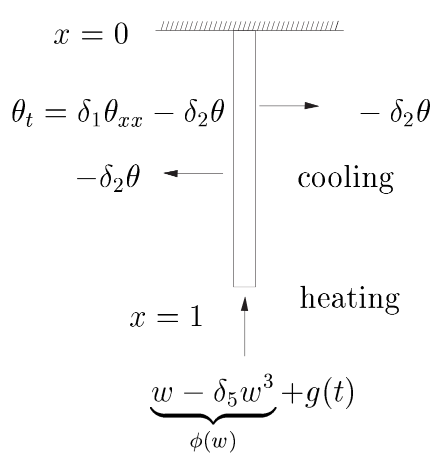
$$\boxed{|g(t)| < \frac{2}{3\sqrt{3\,\delta_5}}} \qquad \text{a.e. } t \in \mathbb{R}\,,$$

$\kappa_2 = \phi'(r_1),\ \kappa_3 = 1 \qquad \Rightarrow$ **(A1)**

$\chi(p) = \dfrac{1}{p+\delta_2}$

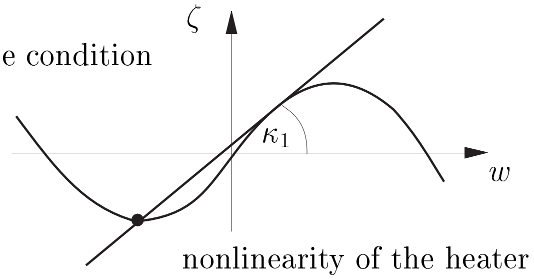
$\lambda \in (0, \delta_2) \qquad \Rightarrow$ **(A3)**

$\lambda^2 - \delta_2\lambda + \kappa_1 \le 0 \qquad \Rightarrow$ **(A6)**

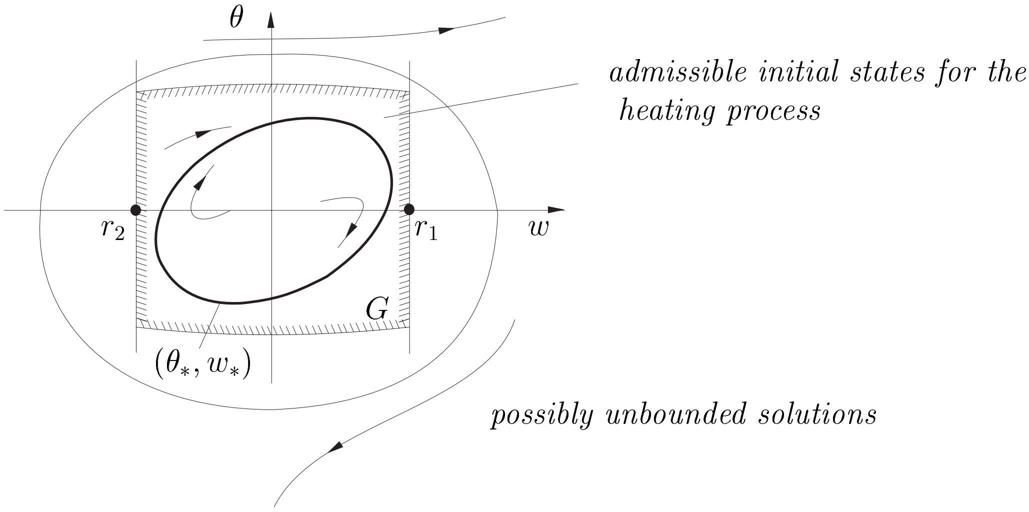


$\boxed{\delta_2^2 \ge 4\,\kappa_1} \qquad \Rightarrow$ **(A3)** + **(A6)** ⇢ cooling condition

⇒ slope condition



Theorem 4.1 $\exists$ *closed, positively invariant and convex set G s.th. inside G there is a unique Bohr a.p. solution (θ_*, w_*) which attracts exponentially for $t \to +\infty$ all other solutions starting in G.*



Localization results

- $w(t) \in [r_2, r_1]$, $\quad t \geq t_0$, $\quad \Rightarrow$ stable heating production process
- $\theta_*(\cdot, t) \in V_1 = W^{1,2}(0,1)$
  $\Rightarrow \quad \int_0^1 \left[\theta_*^2(x,t) + \left(\frac{\partial}{\partial x}\theta_*(x,t)\right)^2\right] dx < \infty\,, \quad t \geq t_0$
  $\Rightarrow$ localized in space variable x solution θ_* ("breather")
- regularity of θ_* $\Rightarrow$ no hot spots