Frequency domain conditions for dynamic buckling in a plate equation with boundary control

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1. Setting of the abstract problem

Suppose Y_0 is a Hilbert space,
$A : \mathcal{D}(A) \to Y_0$ is the generator of a C_0-semigroup on Y_0,
$(\cdot,\cdot)_0, \|\cdot\|_0$ are the scalar product resp. the norm on Y_0
$Y_1 := \mathcal{D}(A)$ with
$(y,\eta)_1 := ((\beta I - A)y, (\beta I - A)\eta)_0,$
$y,\eta \in Y_1,\ \beta \in \rho(A)$ fixed,
$\|\cdot\|_1$ corresponding norm
$Y_{-1} :=$ completion of Y_0 with respect to the norm
$\|y\|_{-1} := \|(\beta I - A)^{-1} y\|_0$,
scalar product
$(y,\eta)_{-1} := ((\beta I - A)^{-1}y, (\beta I - A)^{-1}\eta)_0,$
$y,\eta \in Y_{-1},$
$Y_1 \subset Y_0 \subset Y_{-1}$ densely with continuous embedding, i. e. Gelfand triple
(Y_1, Y_{-1}) is called also Hilbert rigging of the pivot space Y_0;
the Gelfand triple can be extended to a Hilbert scale $\{Y_\alpha\}_{\alpha\in\mathbb{R}}$.
Define the norm in $L^2(0,T;Y_j) \quad (j = 1, 0, -1)$

through $\quad \|y(\cdot)\|_{2,j} := \Big(\int\limits_0^T \|y(t)\|_\alpha^2 dt \Big)^{1/2}.$

Let $\mathcal{L}_T$ denote the space of functions $y : [0, T] \to Y_0$ s.t. $y \in L^2(0, T; Y_{-1})$ and $\dot{y} \in L^2(0, T; Y_{-1})$, where the time derivative $\dot{y}$ is understood in the sense of distributions with values in a Hilbert space. The space $\mathcal{L}_T$ is equipped with the norm

$$\|y\|_{\mathcal{L}_T} := \left(\|y(\cdot)\|_{2,1}^2 + \|\dot{y}(\cdot)\|_{2,-1}^2\right)^{1/2}.$$

Let $y \in Y_0, z \in Y_1$. Then

$$|(y, z)_0| = |(\beta I - A)^{-1} y, (\beta I - A) z)_0| \leq \|y\|_{-1} \|z\|_1.$$

Extending $(\cdot, z)_0$ by continuity onto Y_{-1} we obtain

$|(y, z)_0| \leq \|y\|_{-1} \|z\|_1 \quad \forall y \in Y_{-1}, \forall z \in Y.$

Denote this extension by $(\cdot, \cdot)_{-1,1}$ and call it duality product on $Y_{-1} \times Y_1$.

Consider the control problem

$$\left.\begin{array}{lllll} \dot{y} & = & Ay + Bu & , & u(t) = \varphi(w(t), t), \\ w(t) & = & Cy(t) & , & y(0) = y_0 \quad , \\ z(t) & = & Dy(t) + Eu(t) & , & \end{array}\right\} (1.1)$$

where $A : \mathcal{D}(A) \to Y_0$ is generator of a C_0-semigroup on the Hilbert space $Y_0, B \in \mathcal{L}(U, Y_{-1})$ is the control operator, $C \in \mathcal{L}(Y_1, W)$ is the observation operator, $D \in \mathcal{L}(Y_1, Z)$ and $E \in \mathcal{L}(U, Z)$ are output operators and

$\varphi : W \times \mathbb{R}_+ \to U$ is the nonlinearity.
U, W, Z are Hilbert spaces.

Definition 1.1 $F : Y_1 \to Y_{-1}$ is said to be hemicontinuous if $t \mapsto (F(u+tv), w)_{-1,1}$ is continuous on $[0, 1]$ for all $u, v, w \in Y_1$.
$F : Y_1 \to Y_{-1}$ is said to be monotone if

$$(F(u) - F(v), u - v)_{-1,1} \geq m\|u - v\|_1^2 \qquad \forall u, v \in Y_1.$$

Theorem 1.1. (V. Barbu)

Let $Y_1 \subset Y_0 \subset Y_{-1}$ be a Gelfand triple, and let $F : Y_1 \to Y_{-1}$ be a hemicontinuous monotone operator which satisfies

$$(F(y), y)_{-1,1} \geq \alpha\|y\|_1^2 + \beta \quad \forall y \in Y_1$$

for $\alpha > 0$, and $\beta \in \mathbb{R}$, and

$$\|F(y)\|_{-1} \leq C(\|y\|_1 + 1) \quad \forall y \in Y_1,$$

for $C > 0$. Then, for each $y_0 \in Y_0$ and $g \in L^2(0, T; Y_{-1})$ there exists a unique function y which satisfies
$y \in L^2(0, T; Y_1) \cap C([0, T]; Y_0), \dot{y} \in L^2(0, T; Y_{-1})$,

$$\begin{aligned} \frac{dy}{dt} + F\big(y(t)\big) &= g(t), \text{ a.e. } t \in (0, T), \\ y(0) &= y_0 \,. \end{aligned}$$

2. The frequency theorem

$(\mathbf{H1})$ A is the generator of a stable C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on Y_0, i.e., $\exists M \geq 1, \omega_0 > 0 : \|e^{At}\|_0 \leq Me^{-\omega_0 t} \quad \forall t \geq 0$

$(\mathbf{H2})$ The pair (A, B^*) satisfies the abstract trace regularity condition, i.e., the operator $B^* e^{A^* t}$ admits a continuous extension, denoted by the same symbol, from $Y_0 \to L^2(0, T; U)$:

$$\int_0^T \|B^* e^{A^* t y}\|_U^2 dt \leq c_T \|y\|_0^2 \quad \forall T < \infty, \forall y \in Y_0,$$

where B^* is the dual of B, and $B^* \in \mathcal{L}(\mathcal{D}(A), U)$ (after identifying $[\mathcal{D}(A)]''$ with $\mathcal{D}(A)$.

$(\mathbf{H3})$ $F(y, u) = (F_1\, y, y)_0 + 2\,\mathrm{Re}\,(F_2\, y, u)_U + (F_3 u, u)_U, F_1 \in \mathcal{L}(Y)$, $F_2 \in \mathcal{L}(Y, U)$, $F_3 \in \mathcal{L}(U)$

$(\mathbf{H4})$ $\alpha := \inf\limits_{\omega, y, u} \frac{F(y,u)}{\|y\|_1^2 + \|u\|_U^2} > 0$

where the infimum ranges over all triples $(\omega, y, u) \in \mathbb{R} \times Y_1 \times U$ with $i\omega y = Ay + Bu$

Theorem 2.1 (Frequency theorem for the non-singular case, McMillan, 1997)

Assume the hypotheses $(\mathbf{H1})-(\mathbf{H4})$. Then, there exists an operator $P = P^* \in \mathcal{L}(Y_0)$ s.t.

$2\,\mathrm{Re}(Ay + Bu, Py)_0 + F(y,u) \geq$
$\delta(\|u\|_U^2 + \|y\|_0^2) \qquad \forall (y,u) \in Y_1 \times U$
for some $\delta > 0$

Remark 2.1
a) Instead of $(\mathbf{H2})$ the traditional assumption is the controllability of (A, B).

Definition 2.1 The pair (A, B) is said to be L^2-controllable if, for each $y_0 \in Y_0$, there exists $(y(\cdot), u(\cdot)) \in L^2(\mathbb{R}_+, Y_0) \times L^2(\mathbb{R}_+, U)$ s.t. $y(\cdot)$ is the (weak) solution of
$\dot{y} = Ay + Bu$, $y(0) = y_0$.
In infinite dimension, the L^2-controllability of (A, B) is too restrictive (for instance, if B is compact, the pair (A, B) is never exactly L^2-controllable; Triggiani, 1975)
However, the L^2-controllability of (A, B) holds for some controlled wave equations and systems in which A generates a C_0-group on Y_0 and B is surjective (Curtain, Pritchhard, 1978)

b) Frequency theorem with controllability or regularity condition $(\mathbf{H2})$:

- Yakubovich, 1962, Kalman, 1963; Popov, 1970 - KYP lemma
- Yakubovich, 1974: A, B bounded operators in Hilbert space
- Likhtarnikov, Yakubovich, 1976: A, B unbounded but PDE's on bounded domain and control function in the interior, strong regularity assumptions
- Louis, Wexler, 1991: Control in the interior, removed regularity assumptions
- Lasiecka, Triggiani, 1991; McMillan, 1997 } Frequency theorem for boundary control problems

Example 2.1 Damped Euler-Bernoulli plate equation

$\Omega \subset \mathbb{R}^2$ bounded domain with smooth boundary

$$
\begin{aligned}
w_{tt} + \gamma w_t + \Delta^2 w = 0 \quad & \text{in} \quad \Omega \times (0,T] \quad , \\
\gamma \geq 0 \quad & \\
w(\cdot,0) = w_0,\ w_t(\cdot,0) = w_1 \quad & \text{in} \quad \Omega \\
w_{|\sum} = 0 \quad & \text{in} \quad \partial\Omega \times (0,T] =: \sum \\
\Delta w_{|\sum} = u \quad & \text{in} \quad \sum
\end{aligned}
$$

$u \in L^2(\sum)$ boundary control

$(w_0, w_1) \in V_1 \times V_{-1}$, $V_1 = H_0^1(\Omega)$,

$V_{-1} = H^{-1}(\Omega)$, $V_0 = L^2(\Omega)$

$V_1 \subset V_0 \subset V_{-1}$ Gelfand triple

$y := (w, w_t)$, $Y_0 = V_1 \times V_{-1}$, $U = L^2(\partial\Omega)$

$A_0 h := \Delta^2 h$,

$\mathcal{D}(A_0) = \{h \in H^4(\Omega) : h_{|\partial\Omega} = \Delta h_{|\partial\Omega} = 0\}$

$$A := \begin{bmatrix} 0 & I \\ -A_0 & -\gamma I \end{bmatrix}, \; Bu := \begin{bmatrix} 0 \\ A_0 \mathcal{G} u \end{bmatrix},$$

$F_1 = I, F_2 = F_3 = 0$

$\mathcal{G}$ is the Green map defined by

$h = \mathcal{G}v \Leftrightarrow \{\Delta^2 h = 0, h_{|\partial\Omega} = 0, \Delta h_{|\partial\Omega} = 0\}$

A is stable on Y_0 $\Rightarrow$ $(\mathbf{H1})$

Lasiecka / Triggiani, 1991:

$A^{-1}B \in \mathcal{L}(U, Y_0) = \Phi(t; \Phi_0, \Phi_1)$

and $B^* e^{A^* t}\binom{w}{w_t} = \frac{\partial \Delta \Phi(t)}{\partial \nu}, (w, w_t) \in Y_0,$

Φ solution of the associated homogeneous problem

$$\int_{\sum} \left|\frac{\partial \Delta \Phi}{\partial \nu}\right|^2 d\sum \leq C_T \|(\Phi_0, \Phi_1)\|_{Y_0}^2$$

$\Rightarrow$ McMillan's frequency theorem is applicable

Example 2.2 (Likhtarnikov / Yakubovich, 1976)

$\Omega \subset \mathbb{R}^n$ domain with smooth boundary
$Y_0 = L^2(\Omega)\,,\ Y_1 = W^{1,2}(\Omega)\,,\ Y_{-1} \cong Y_1'\,,$
$U = W^{-1/2,2}(\partial\Omega)$
$\Rightarrow Y_1 \subset Y_0 \subset Y_{-1}$ Gelfand triple

$$A : \mathcal{D}(A) \to Y_0,\ a(w,z) := \int_\Omega \sum_{i=1}^n w_{x_i}\,\bar{z}_{x_i}\,d\Omega\,,$$
$$w, z \in W^{1,2}(\Omega)$$

$$B \in \mathcal{L}(U, Y_{-1})\ : b\,(u,w) = \int_{\partial\Omega} u(x)\,\overline{w(x)}\,dS$$
$$w \in W^{1,2}(\Omega)\,,\ u \in W^{-1/2,\,2}(\partial\Omega)$$

$$\Rightarrow \dot{y} = Ay + Bu \quad \text{in } Y_0$$
$$y(0) = y_0 \in Y_0 \qquad (2.1)$$

For smooth data and smooth region (2.1) is equivalent to the boundary control problem

$$w_t = \Delta w + f \quad \text{in} \quad \Omega \times (0, +\infty)$$
$$\frac{\partial w}{\partial \nu}_{|\sum} = u(x,t) \quad \text{in} \quad \partial\Omega \times (0,+\infty) =: \sum$$
$$w(x,0) = w_0(x) \quad \text{in} \quad \Omega$$

3. Absolute stability and instability

Definition 3.1

a) We say that a pair $\{w(\cdot), u(\cdot)\} \in L^2(0,\infty;W)\times L^2(0,\infty;U)$ belongs to $\mathcal{M}(F)$ if $F(w(t),u(t)) \leq 0$ for a.e. $t \geq 0$.

The class of nonlinearities defined by F is $\mathcal{N}(F) := \{\varphi : W \times \mathbb{R}_+ \to U$ s.t. for any $w(\cdot) \in L^2(0,\infty;W)$ follows $\{w(\cdot), \varphi(w(\cdot))\} \in \mathcal{M}(F)\}$

b) The nonlinear system (1.1) is said to be absolutely stable with respect to the output w in the class $\mathcal{N}(F)$ if for any triple $\{y,w,u\}$ s.t. $\dot{y} = Ay + Bu$, $w = Cy$ and $\{w,u\} \in \mathcal{M}(F)$ we have

$$\int_0^\infty \|w\|_W^2 dt \leq C_1 \|w(0)\|_W^2 + C_2$$

$(C_1$ and C_2 depend only on $\mathcal{N}(F))$.

c) The nonlinear system (1.1) is said to be absolutely unstable in the class $\mathcal{N}(F)$ if for any $\varphi \in \mathcal{N}(F)$ the associated system (1.1) has solutions y with $y(\cdot) \notin L^2(0,\infty;Y_0)$.

Theorem 3.1 Assume that the following conditions are satsfied:

1) A is the generator of a stable C_0-semigroup;
2) The pair (A, B^*) satisfies the trace property;
3) $\exists \delta > 0 : F(\mathcal{X}(i\omega)u, u) \geq \delta \|\mathcal{X}(i\omega)u\|_W^2$
$\forall u \in U \quad \forall \omega \in \mathbb{R} : i\omega \notin \sigma(A)$.
Then (1.1) is absolutely stable in the class $\mathcal{N}(F)$.

Theorem 3.2 (Likhtarnikov, 1979) Assume:
1) A is the generator of a stable C_0-semigroup $\{e^{At}\}_{t\geq 0}$;
2) $\{e^{At}\}_{t\geq 0}$ is extentable to a group on $\mathbb{R}$;
3) The pair $(-A, B)$ is L^2-controllable;
4) The frequency domain condition from Theorem 3.1 is satisfied;
Then (1.1) is absolutely stable in the class $\mathcal{N}(F)$.

Definition 3.2 We say that $A : \mathcal{D}(A) \to Y_0$ is the generator of an unstable C_0-semigroup on Y_0 if A generates a semigroup $\{e^{At}\}_{t\geq 0}$ and $\omega(A) := \lim_{t\to\infty} \frac{\ln \|e^{At}\|_0}{t} > 0$, where $\omega(A)$ is the growth bound.

Remark 3.1 For a C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on the Hilbert space Y_0 let

$$s(A) := \sup\{\mathrm{Re}\, s : s \in \sigma(A)\}$$

be the spectral bound of A.

Under certain assumptions on A (for instance, if A is generator of an analytic semigroup) we have $\omega(A) = s(A)$.
However, generally we have only $\omega(A) \geq s(A)$.
For the two-dimensional wave equation is $\omega(A) > s(A)$ (Renardy, 1994).

Theorem 3.3 Suppose that the following conditions are satisfied:
1) $A : \mathcal{D}(A) \to X_0$ is the generator of an unstable C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on Y_0;
2) The pair (A, B^*) satisfies the trace property;
3) Frequency domain condition 3) from Theorem 3.1 ;
Then (1.1) is absolutely unstable in the class $\mathcal{N}(F)$.

4. Stability analysis of PDE's on the base of measurements

Consider the parameter-dependent problem

$$\left.\begin{array}{l} \dot{y} = A(q)y + B(q)u \quad , \quad u(t) = \varphi(w(t)), \\ w(t) = C(q)y, \quad z(t) = D(q)y + E(q)u \end{array}\right\} (4.1)_q$$

Q a metric space with metric d

For any $q \in Q$ we suppose
$A(q) : \mathcal{D}(A(q)) \to Y_0$ is generator of a C_0-semigroup on Y_0,
$B(q) \in \mathcal{L}(U, Y_{-1}), C(q) \in \mathcal{L}(Y_1, W)$,
$D(q) \in \mathcal{L}(Y_1, Z)$,
$E(q) \in \mathcal{L}(U, Z)$

$$\left.\begin{array}{lcl} \mathcal{X}^{(w)}(s,q) & = & C(q)\Big(sI - A(q)\Big)^{-1} B(q) \\ \mathcal{X}^{(z)}(s,q) & = & D(q)\Big(sI - A(q)\Big)^{-1} B(q) + E(q) \end{array}\right\} \text{transfer operators}$$

$\varphi : W \times \mathbb{R}^1 \to U$,
$\varphi \in \mathcal{N}(q) := \{\Psi : W \times \mathbb{R}^1 \to U$,

$F(w(t), \varphi(w(t)), q) \le 0$, $t \in [0, T], \forall w(\cdot) \in L^2(0, T; W\}$,

$F(w, u, q) = (F_1(q)w, w)_W + 2\mathrm{Re}(F_2(q)w, u)_U + (F_3(q)u, u)_U$

$F_1(q) = F_1(q)^* \in \mathcal{L}(W), F_2(q) \in \mathcal{L}(W, U)$,

$$F_3(q) = F_3(q)^* \in \mathcal{L}(U)$$

$J_\nu(\cdot,\cdot) : Q \times \mathcal{T} \to \mathbb{R}\,, \nu = 1, 2, \ldots, k,$
stability functionals
$\mathcal{T}$ Hilbert space , $J = (J_1, \ldots, J_k) \in \mathcal{S}$

$\tilde{Q}(\tau) := \{q \in Q : J_\nu(q, \tau) \leq 0, \nu = 1, 2, \ldots, k\}$

$Q_{abs} \subset Q$ is the set of all $q \in Q$ s.t.$(4.1)_q$ is absolute stable with respect to the output $z(\cdot)$ in the class $\mathcal{N}(q)$

$$\Leftrightarrow \exists \tau_{abs} \in \mathcal{T} \text{ s.t. } Q_{abs} = \tilde{Q}(\tau_{abs})$$

$$Z^N(t) = D^N y(t) + E^N u(t) \qquad (4.2)_N$$

$D^N : Y \to Z^N, E^N : U \to Z^N$
$Z^N \subset Z, \mathcal{T}^M \subset \mathcal{T}$ finite dimensional subspaces
$\tilde{Q}(\tau^M) = \{q \in Q : J_\nu(q, \tau^M) \leq 0, \nu = 1, 2, \ldots, k\}$

$Q_{abs}(N) \subset Q$ is the set of all $q \in Q$ s.t. $(4.1)_q, (4.2)_N$ is absolutely stable with respect to the output $Z^N(\cdot)$ in the class $\mathcal{N}(q) \Leftrightarrow$
$\exists\, M\, \exists \tau^M_{abs} \in \mathcal{T}^M$ s.t. $Q_{abs}(N) = \tilde{Q}(\tau^M)$.

Theorem 4.1 Suppose that $\tau^M_{abs} \to \tau$ for $M \to \infty$ in $\mathcal{T}$. Then $\tilde{Q}(\tau) = Q_{abs}$.