1. Gelfand triples and solution spaces

Suppose Y_0 is a Hilbert space,
$(\cdot,\cdot)_0, \|\cdot\|_0$ are the scalar product resp. the norm on Y_0
$A : \mathcal{D}(A) \to Y_0$ is the generator of a C_0-semigroup on Y_0,
$Y_1 := \mathcal{D}(A)$ with
$(y,\eta)_1 := ((\beta I - A)y, (\beta I - A)\eta)_0, y, \eta \in Y_1,$
$\beta \in \rho(A)$ fixed,
$\|\cdot\|_1$ corresponding norm
$Y_{-1} :=$ completion of Y_0 with respect to the norm
$\|y\|_{-1} := \|(\beta I - A)^{-1} y\|_0$,
associated scalar product
$(y,\eta)_{-1} := ((\beta I - A)^{-1} y, (\beta I - A)^{-1}\eta)_0,$
$y, \eta \in Y_{-1},$
$\Rightarrow Y_1 \subset Y_0 \subset Y_{-1}$ densely with continuous embedding
$(Y_\alpha \subset Y_{\alpha-1}, \alpha = 1, 0,$ dense and
$\|y\|_{\alpha-1} \le C\|y\|_\alpha, \forall y \in Y_\alpha)$, i. e. Gelfand triple
(Y_1, Y_{-1}) is also called Hilbert rigging of the pivot space Y_0;
Y_1 is the interpolation space,
Y_{-1} is the extrapolation space, the Gelfand triple can be extended to a
Hilbert scale $\{Y_\alpha\}_{\alpha\in\mathbb{R}}$.
Let $y \in Y_0, z \in Y_1$. Then

$$|(y,z)_0| = |(\beta I - A)^{-1} y, (\beta I - A)z)_0| \le \|y\|_{-1}\|z\|_1.$$

Extending $(\cdot, z)_0$ by continuity onto Y_{-1} we obtain
$|(y,z)_0| \le \|y\|_{-1}\|z\|_1 \ \forall y \in Y_{-1}, \forall z \in Y_1.$
Denote this extension by $(\cdot,\cdot)_{-1,1}$ and call it
duality product on $Y_{-1} \times Y_1$.
Suppose $T > 0$ arbitrary and define the norm in $L^2(0,T;Y_j)$
$(j = 1, 0, -1)$

through $$\|y(\cdot)\|_{2,j} := \Big(\int_0^T \|y(t)\|_\alpha^2 dt\Big)^{1/2}.$$

Let $\mathcal{L}_T$ denote the space of functions $y : [0,T] \to Y_0$ s.t. $y \in L^2(0,T;Y_1)$ and $\dot{y} \in L^2(0,T;Y_{-1})$, where the time derivative $\dot{y}$ is understood in the sense of distributions with values in a Hilbert space.
The space $\mathcal{L}_T$ (solution space) equipped with the norm

$$\|y\|_{\mathcal{L}_T} := \left(\|y(\cdot)\|_{2,1}^2 + \|\dot{y}(\cdot)\|_{2,-1}^2\right)^{1/2} \text{ is a Hilbert space,}$$

Remark 1.1
Denote by $C(0,T;Y_0) =: C_T$ the Banach space of continuous mappings $y : [0,T] \to Y_0$ provided with the norm
$\|y(\cdot)\|_{C_T} = \sup_{t\in[0,T]} \|y(t)\|_0$

$\mathcal{L}_T$ can be continuously imbedded into the space C_T, i.e., every function from $\mathcal{L}_T$, properly altered by some set of measure zero, is a continuous function $y : [0,T] \to Y_0$ and

$$\|y(\cdot)\|_{C_T} \leq \text{const} \cdot \|y(\cdot)\|_{\mathcal{L}_T}.$$

Example 1.1

$$\begin{aligned} Y_0 &= L^2(\mathbb{R}_+,\mathbb{R}^m) \quad \text{pivot space} \\ \text{interpolation space } Y_1 &= \{f \in L^2(\mathbb{R}_+,\mathbb{R}^m), \text{supp}\, f \text{ compact}\} \\ \text{supp}\, f &= \overline{\{x \in \mathbb{R}_+ : f(x) \neq 0\}} \\ Y_1 &\subset Y_0 \quad \text{dense} \end{aligned}$$

extrapolation space $Y_{-1} :=$ closure of Y_0 w.r.t. Y_1
$\Rightarrow Y_{-1} = L^2_{loc}(\mathbb{R}_+,\mathbb{R}^m)$
$\Rightarrow$ any Wiener process can be considered as element of Y_{-1}

2. Evolutionary variational inequalities

Consider the observed and controlled evolutionary variational inequality (OCEVI)

$$\left.\begin{array}{l} (\dot{y} - Ay - B\xi, y - \eta)_{-1,1} + \psi(\eta) - \psi(y) \geq 0 \\ y(0) = y_0 \in Y_0\ ,\ \forall \eta \in Y_1\ ,\ \text{a.e. } t \in [0,T], \\ \xi(t) \in \varphi(t, w(t)) \quad \text{a.e. } t \in [0,T] \quad \underline{\text{control}}, \\ w(t) = Cy(t) \qquad \underline{\text{output}}\,, \\ z(t) = Dy(t) + E\xi(t) \quad \underline{\text{observation}} \end{array}\right\} (2.1)$$

where $A : \mathcal{D}(A) \to Y_0$ is generator of a C_0-semigroup on the Hilbert space Y_0, $B : \Xi \to Y_{-1}$ (control operator), $D : Y_1 \to Z$ and $E : \Xi \to Z$ (observation operators) are linear bounded operators, Ξ (control space), W (output space) and Z (observation space) are Hilbert spaces,
$\varphi : \mathbb{R}_+ \times W \to 2^{\Xi}$ (material law map) and
$\psi : Y_1 \to \mathbb{R}_+$ (contact functional) are in general nonlinear,

Definition 2.1 Any function $y \in \mathcal{L}_T$ satisfying (2.1) is called a solution of (2.1)

(A1) Problem (2.1) is well-posed on any compact interval $[0,T]$, i.e., for arbitrary $y_0 \in Y_0$ there exists a unique function $y(\cdot) \in \mathcal{L}_T$ satisfying (2.1) and depending continuously on the initial data y_0 and φ.

Special case:
Observed and controlled evolutionary variational equality (OCEVE)
$\psi \equiv 0 \Rightarrow$
$(\dot{y} - Ay - B\xi, y - \eta)_{-1,1} = 0 \qquad \forall \eta \in Y_0$
$\Leftrightarrow$

$$\left.\begin{array}{rcl} \dot{y} & = & Ay + B\xi \quad \text{in } Y_{-1} \\ \xi(t) & \in & \varphi(t, w(t)),\ w(t) = Cy(t), \\ z(t) & = & Dy(t) + E\xi(t) \end{array}\right\} (2.2)$$

Let y be the solution of $\dot{y} = Ay, y(0) = y_0 \in Y_0$
Define the operator $e^{At}y_0 := y(t) \in Y_0$
a. e. $t \in [0, T]$.

$\Rightarrow$ a) $e^{At} : Y_0 \to Y_0$;

b) $t \mapsto e^{At}y_0$ is continuous in the Y_0- norm ;

c) $e^{A0} = I_0$, where I_0 is the identity operator in Y_0;

d) $e^{A(t+s)} = e^{At}e^{As} = e^{As}e^{At}$, $t, s \in [0, T]$

Definition 2.2

a) Suppose F is a quadratic form on $W \times \Xi$
The class of nonlinearities $\mathcal{N}(F)$ defined by F consists of all maps $\varphi : \mathbb{R}_+ \times W \to 2^{\Xi}$
s. t. for any $y(\cdot) \in L^2_{loc}(0, \infty; Y_1)$ with $\dot{y}(\cdot) \in L^2_{loc}(0, \infty; Y_{-1})$ and any $\xi(\cdot) \in L^2_{loc}(0, \infty; \Xi)$ with $\xi(t) \in \varphi(t, Cy(t))$ for a. e. $t \geq 0$, it follows that $F(w(t), \xi(t)) \geq 0$ for a.e. $t \geq 0$.

b) The class of functionals $\mathcal{M}(d)$ defined by a constant $d > 0$ consists of all maps $\psi : Y_1 \to \mathbb{R}_+$ s.t. $t \mapsto \psi(y(t))$ belongs for any $y \in L^2_{loc}(0, \infty; Y_0)$ with $\dot{y} \in L^2_{loc}(0, \infty; Y_1)$ to $L^1(0, \infty; \mathbb{R})$ satisfying $\int\limits_0^\infty \psi(y(t))dt \leq d$ and for any $\varphi \in \mathcal{N}(F)$ and any $\psi \in \mathcal{M}(d)$ the inequality (2.1) is well-defined on any time interval $[0, T]$.

c) Any triple of functions (y, ξ, ψ) is called a response of (2.1) w.r.t. the classes $\mathcal{N}(F)$ and $\mathcal{M}(d)$ if y is together with $\xi(t) \in \varphi(t, Cy(t))$ solution of (2.1) for the given $\psi \in \mathcal{M}(d)$

Example 2.1 (Likhtarnikov/Yakubovich, 2000)
$\Omega \subset \mathbb{R}^n$ bounded domain, $\partial\Omega$ smooth

$$\left.\begin{array}{rcl} u_{tt} + 2\varepsilon u_t - \Delta u + \alpha u & = & f(\Theta) \\ \Theta_t - \beta\Delta\Theta + u - \gamma g(\Theta) & = & 0 \end{array}\right\} \quad (2.3)$$

$\varepsilon > 0,\ \beta > 0, \alpha,\ \gamma$ real parameters,
u deflection, Θ temperature

membrane equation of nonlinear thermo-elasticity

IC : $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$,
$\Theta(x,0) = \Theta_0(x)$
BC: $u(x,t) = \Theta(x,t) = 0$, $x \in \partial\Omega$

Class of nonlinearities:
$\Theta g(\Theta) - f^2(\Theta) \geq 0 \qquad \forall\Theta \in \mathbb{R}$ (2.4)
e.g., $f(\Theta) = \Theta^2, g(\Theta) = \Theta^3$.

$$y(x,t) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} u_t \\ u \\ \Theta \end{bmatrix},$$

$$\varphi = \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \xi$$

- $A_0 = A_0^*$ positive operator generated in $L^2(\Omega)$ by $(-\Delta)$ (with zero boundary cond.)
  $D(A_0) = W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$
- $V_s := D(A_0^{s/2})$, $s \in \mathbb{R}$, with the scalar product

$(u,v)_s := (A_0^{s/2}u, A_0^{s/2}v) \leftarrow$ scalar product in $L_2(\Omega)$ is a Hilbert scale

- $\Xi := L^2(\Omega) \times L^2(\Omega)$ - control space
- $Y_0 := V_0 \times V_1 \times V_1$ - pivot space
- $Y_1 := V_1 \times V_1 \times V_2$ - interpolation space
- $Y_{-1} \cong Y_1^*$ - extrapolation space

scalar product in Y_s :
$(y,z)_s = (y_1,z_1)_{s-1} + (y_2,z_2)_s$.

$$A = \left[\begin{array}{ccc} -2\varepsilon I & -A_0 - \alpha I & 0 \\ 1 & 0 & 0 \\ 0 & -I & -\beta A_0 \end{array} \right] -$$ generator of a C_0 semigroup

$$B = \left[\begin{array}{cc} -I & 0 \\ 0 & 0 \\ 0 & -\gamma I \end{array} \right]$$ – control operator

$\Rightarrow (A, B)$ is L^2-controllable
(stabilizable with $\xi_1 = \alpha y_1, \xi_2 = 0$)

- Quadratic constraint:

$$F(y, \xi) = \int_\Omega (y_3 \xi_2 - \xi_1^2) dx =$$

$$= \int_\Omega \underbrace{[\Theta(x) g(\Theta(x)) - f^2(\Theta(x))]}_{\geq 0} dx$$

$\Rightarrow F(y, \xi) \geq 0$ for all $y \in Y_0$ and nonlinearities ξ satisfying (2.4)

- Observations:

$$w = (u_t, u, \Theta) \qquad (D = I_0,\ E = 0)$$

or $$w = (u_{tt}, u_t, \Theta_t) = w_t \quad (D = \frac{\partial}{\partial t},\ E = 0)$$

Example 2.2 Final state estimator (without noise)
Given

$$\begin{array}{rcll} \dot{y}(t) & = & Ay(t) + \xi(t) & , \quad y(-\infty) = 0 \\ w(t) & = & Cy(t) & , \qquad t \leq 0 \\ \xi & \in & L^2((-\infty, 0], Y_0) & \text{with compact support.} \end{array}$$

The system is at rest before ξ becomes active, i.e. $y(t) = 0$ if $\xi(\tau) = 0$ for all $\tau \leq t$.

The final state estimation problem for (A, C) is to find a bounded linear operator

$$E : L^2((-\infty, 0], W) \to Z$$
$$z(t) = E\, w(t) = E\, Cy(t)$$
$$\text{s. t. } \sup_{\|\xi\| \le 1} \|E\, w(t) - y(0)\| < +\infty.$$

$\Rightarrow \quad E$: Kalman estimator
$z(t) = A\, z(t) + (-CP)(C\, z(t) - w(t))$
Here $P = P^* : Y_0 \to Y_0$ is a solution of the Riccati equation $A^*P + PA - PP + I = 0$.

Example 2.3 $u_{tt} + \gamma u_t + \Delta u + \frac{\partial}{\partial x}\left(g(\frac{\partial}{\partial x} u)\right) = 0$
$u(0, t) = u(l, t) = 0 \quad , t > 0$
$u(x, t) = u_0(x)\ ,\ u_t(x, t) = u_1(x)\ , t > 0, x \in (0, l)$
Point observation operator C:
Y_0 state space of $y(t) = [u(\cdot, t), u_t(\cdot, t)]$
$z(t) = Cy(t) := [u(\alpha_1, t), \ldots, u(\alpha_j, t), u_t(\beta_1, t), \ldots, u_t(\beta_k, t)] \in \mathbb{R}^{j+k}$
$\Rightarrow C$ is a bounded operator on Y_0

Example 2.4 Consider the general inequality (2.1) with $Z = Y_{-1}$ and $\psi = 0$. Define the observation map by $\{y(\cdot), \xi(\cdot)\} \in L^2(0, \infty; Y_1) \times L^2(0, \infty; \Xi) \mapsto z(\cdot) := Ay(\cdot) + B\xi(\cdot) \in Y_{-1}$.
$\Rightarrow z(t) = \dot{y}(t)$ Observation of the velocity

3. The Frequency Domain Theorem

(**A 1**) The operator $A \in \mathcal{L}(Y_0, Y_{-1})$ is regular, i.e. for any $T > 0, y_0 \in Y_1, \psi_T \in Y_1$ and $f \in L^2(0, T; Y_0)$ the solutions of the direct problem

$$\dot{y} = Ay + f(t), y(0) = y_0,\ t \in [0, T]$$

and of the dual problem

$$\dot{\psi} = -A^*\psi, \psi(T) = \psi_T\,,\ t \in [0, T]$$

are strongly continuous in the norm of Y_1.

Remark 3.1 The condition is satisfied if the imbedding $Y_1 \subset Y_0$ is completely continuous, i. e. transforms bounded sets from Y_1 into compact sets in Y_0.

(**A 2**) The pair (A, B) is L^2-controllable, i.e., there exists an operator $K \in \mathcal{L}(Y_1, \Xi)$ such that the problem

$$\dot{y} = (A + BK)y \ , \quad y(0) = y_0$$

is well-posed on the semiaxis $[0, +\infty)$.

(**A3**) Let $F(y, \xi)$ be a Hermitian form on $Y_1 \times \Xi$,

$$F(y, \xi) = (F_1 y, y)_{-1,1} + 2\text{Re}(F_2 y, \xi)_\Xi + (F_3 \xi, \xi)_\Xi,$$

where

$$F_1 = F_1^* \in \mathcal{L}(Y_1, Y_{-1}) \ , \ F_2 \in \mathcal{L}(\Xi, Y_0) \ , \ F_3 = F_3 * \in \mathcal{L}(\Xi, \Xi)$$

Define

$$\alpha := \sup_{\omega, y, \xi} (\|y\|_1^2 + \|\xi\|_\Xi^2)^{-1} F(y, \xi) \ ,$$

where the infimum is taken over all triples $(\omega, y, \xi) \in \mathbb{R}_+ \times Y_1 \times \Xi$ such that $i\omega y = Ay + B\xi$, and assume $\alpha < 0$ (Frequency-domain condition).

Theorem 3.1 (Frequency Theorem for the Nonsingular Case) Assume that $A \in \mathcal{L}(Y_1, Y_{-1}), B \in \mathcal{L}(\Xi, Y_{-1})$ and the Hermitian form F on $Y_1 \times \Xi$ satisfy the assumption (**A 1**) - (**A 3**). Then there exist an operator $P = P^* \in \mathcal{L}(Y_0, Y_0)$ and a number $\delta > 0$ such that

$$\begin{aligned} &\text{Re}(Ay + B\xi, Py)_0 + F(y, \xi) \leq \\ &\leq -\delta(\|y\|_1^2 + \|\xi\|_\Xi^2), \forall (y, \xi) \in Y_1 \times \Xi \end{aligned} \qquad (3.1)$$

Proof: Likhtarnikov / Yakubovich, 1976.

Corollary 3.1 Under the assumptions of Theorem 3.1 there exist an operator $P = P^* \in \mathcal{L}(Y_0, Y_0)$ and a number $\delta > 0$ s. t. the form $\mathcal{V}(y) := (y, Py)_0 \quad (y \in Y_0)$ satisfies for any solution $y(\cdot)$ of (2.1) the inequality

$$\mathcal{V}(y(t))-\mathcal{V}(y(s))+\int_s^t F(y(\tau),\xi(\tau))d\tau+\int_s^t (\psi(y(\tau)-$$
$$-\psi(-Py(\tau)+y(\tau))]d\tau+\delta\int_s^t \|z(\tau)\|_z^2 d\tau \le 0. \quad (3.2)$$

Remark 3.2 For $\psi = 0$ ineq. (3.2) is called dissipation inequality: It can be considered as generalized energy balance inequality with the energy storage function $\mathcal{V}$, the energy supply rate term given by F (influence of the constitutive law), a contact energy term characterized through P, and a dissipation rate term depending on δ.

4. Absolute observation-stability of evolutionary inequalities

Definition 4.1 The inequality (2.1) is said to be absolutely observation-dichotomic if for any admissible response $\{y,\xi,\psi\}$ of (2.1) with $y(0)=y_0$ and $y(\cdot)$ bounded on $[0,\infty)$ in Y_0 it follows that

$$\|z(\cdot)\|_{2,Z}^2 \le C_1(\|Y_0\|_0^2 + C_2)\ , \quad (4.1)$$

where the constants C_1 and C_2 depend only on $A, B, \mathcal{N}(F)$ and $\mathcal{M}(d)$.
The inequality (2.1) is said to be absolutely observation-stable if (4.1) holds for any admissible.

Definition 4.2 For $s \in \mathbb{C}\backslash\rho(A)$ define the transfer operator of (2.1) w. r. t. the control w by

$$\chi^{(w)}(s) = C(sI-A)^{-1}B$$

and the transfer operator of (2.1) w. r. t. the observation z by

$$\chi^{(z)}(s) = D(sI-A)^{-1}B + E.$$

(**A4**) There exists a $\delta > 0$ s. t.

$$F((i\omega I-A)^{-1}B\xi,\xi) \ge \delta\|\chi^{(z)}(i\omega)\xi\|_Z^2$$
$$\forall i\omega \notin \sigma(A)\ ,\ \forall \xi \in \Xi.$$

Theorem 4.1 Suppose that the assumptions (**A1**), (**A2**) and (**A4**) are satisfied. Then inequality (2.1) is absolutely observation-dichotomic.

Definition 4.3 The inequality (2.1) is said to be minimally stable if the resulting equality for $\eta = 0$ is minimally stable, i.e., there exists a bounded linear operator $K : Y_1 \to \Xi$ s. t. the operator $A + BK$ is stable

$$(\sigma(A+BK) \subset \{z \in \mathbb{C} : \mathrm{Re}\, z \leq -\varepsilon < 0\}) \text{ and}$$
$$F(y, Ky) \geq 0 \quad \forall y \in Y_1.$$

Theorem 4.2 Suppose that the assumptions (**A1**), (**A2**) and (**A4**) are satisfied and the inequality (2.1) is minimally stable. Then this inequality is absolutely observation-stable.

Example 4.1 Beam equation with Hookean material

$$\rho A \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\frac{EA}{3}\tilde{\varphi}\left(\frac{\partial u}{\partial x}\right)\right) = 0$$

$u(0,t) = u(l,t) = 0\ ,\ t > 0$
$u(x,0) = u_0(x)\ ,\ u_t(x,0) = u_1(x)\ ,\ x \in (0,l)$
$\tilde{\varphi}(w) = 1 + w - (1+w)^{-2} \quad w \in (-1,1)$
Break the stress-strain law $\tilde{\varphi}$ into the sum of a linear term and a nonlinear term φ:
$\rho A \frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\frac{EA}{3}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{EA}{3}\varphi\left(\frac{\partial u}{\partial x}\right)\right) = 0$
$\Rightarrow\ u_{tt} + 2\varepsilon u_t - \alpha u_{xx} = -\alpha(-\frac{\partial}{\partial x}\varphi(\frac{\partial u}{\partial x})) =: \alpha\frac{\partial}{\partial x}\xi$
$\lambda_k > 0\, ,\, e_k\, ,\, k = 1, 2, \ldots$, eigenvalues and eigenfunctions of the operator $(-\Delta)$ with zero boundary conditions
Fourier series (formally): $u(x,t) = \sum_k u^k(t)e_k$, $\xi(x,t) = \sum_k \xi^k(t)e_k$
$+$ Fourier transformation:
$-\omega^2\tilde{u}^k(i\omega) + 2i\omega\varepsilon\tilde{u}^k(i\omega) + \alpha\lambda_k\tilde{u}^k(i\omega) = -\alpha\sqrt{\lambda}_k\tilde{\xi}^j(t)$
$\Rightarrow \tilde{u}^k = \chi(i\omega, \lambda_k)\tilde{\xi}^k$,
$\chi(i\omega, \lambda_k) = (-\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k)^{-1}(\alpha\sqrt{\lambda k})$,
$k = 1, 2, \ldots$.

Functional for the nonlinearity $\varphi \in \mathcal{N}(F)$,
$F(w,\xi) = \kappa w^2 - \xi w$,
$\mathcal{J}(w,\xi) = \text{Re} \int_0^\infty (\kappa |w|^2 - w\bar{\xi}) dx dt =$
$\text{Re} \int_{-\infty}^{+\infty} (\kappa |\tilde{w}|^2 - \tilde{w}\bar{\tilde{\xi}}) dt.$
$|\tilde{w}|^2 = \sum_k \lambda_k |\tilde{u}^k|^2 = \sum_k \lambda_k |\chi(i\omega, \lambda_k)|^2 |\tilde{\xi}^k|^2$
$\tilde{w}\bar{\tilde{\xi}} = \sum_k \sqrt{\lambda_k}\, \tilde{u}^k \overline{\tilde{\xi}^k} = \sum_k \sqrt{\lambda_k}\, \chi(i\omega, \lambda_k) |\tilde{\xi}^k|^2$
$\Rightarrow \mathcal{J} = \text{Re} \int_{-\infty}^{+\infty} \Bigg[\kappa \Bigg(\sum_k \lambda_k |\chi(i\omega, \lambda_k)|^2 |\tilde{\xi}^k|^2 \Bigg)$
$- \sum_k \sqrt{\lambda_k} \chi(i\omega, \lambda_k) |\tilde{\xi}^k|^2 \Bigg] dt$
$\Rightarrow \prod_0^k (i\omega) = \kappa \lambda_k |\chi(i\omega, \lambda_k)|^2 - \sqrt{\lambda_k} \text{Re}\, \chi(i\omega, \lambda_k) < 0,$
$\forall \omega \in \mathbb{R},\ k = 1, 2, \dots .$
$\Leftrightarrow \text{Re}\, \chi(i\omega, \lambda_k) - \kappa \sqrt{\lambda_k} |\chi(i\omega, \lambda_k)|^2 > 0$
$\forall \omega \in \mathbb{R},\ k = 1, 2, \dots .$
$\chi(i\omega, \lambda_k) = (-\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k)^{-1} (-\alpha \sqrt{\lambda_k})$
$\text{Re}\, \chi(i\omega, \lambda_k) = [(\alpha\lambda_k - \omega^2)^2 + 4\omega^2\varepsilon^2]^{-1} \kappa \sqrt{\lambda_k} (\omega^2 - \alpha\lambda_k)$

5. Global asymptotics of autonomous inequalities

Definition 5.1 Consider the autonomous inequality (2.1) $(\varphi(t,w) \equiv \varphi(w))$. A solution $y(\cdot)$ of (2.1) is called stationary if $\dot{y}(t) = 0$ for a. e. $t \geq 0$. The set $\Lambda = \{y(\cdot)$ stationary solution of (2.1)$\}$ is called the stationary set of (2.1). For any solution $y(\cdot)$ with initial point y_0 of (2.1) $\gamma^t(y_0) = \{y(t), t \geq 0\}$ is an orbit through y_0. The solution $y(\cdot)$ is called bounded if its orbit is bounded and compact if its orbit is contained in a compact set in Y_0. The autonomous inequality (2.1) is called dichotomic if any its bounded orbit tends to the stationary set Λ for $t \to +\infty$. The autonomous inequality is said to be dissipative if in Y_0 there exists a bounded absorbing set $\mathcal{B}_0$ s. t. for any bounded set.

$B \subset Y_0$ there exists a $t_0 > 0$ s. t. $y(t, y_0) \in \mathcal{B}_0$ for all $t \geq t_0$ and all $y_0 \in \mathcal{B}_0$. The inequality is called compactly dissipative if it is dissipative with a compact absorbing set. The inequality has a global asymptotics if the orbit of any its solution tends to Λ for $t \to \infty$.

Notation: Suppose Λ_i is a connected component of Λ and $W^u(\Lambda_i)$ is the unstable manifold of Λ_i,
i.e., $W^u(\Lambda_i) = \{y(\cdot)$ solution of (2.1): $\exists t_n \to -\infty$ with $y(t_n) \to \Lambda_i$ for $n \to +\infty\}$.
(For $y \in W^u(\Lambda_i)$ it is assumed that there exist solutions also for $t \to -\infty$.)

Definition 5.2 A global attractor A of (2.1) is called quasi-regular if $A = \bigcup\limits_i W^u(\Lambda_i)$.

Theorem 5.1 Consider the autonomous inequality (2.1) and assume that A is a global attractor of (2.1). Suppose that the inequality is absolutely observation-stable w.r.t. the observation operator $z = Ay + B\xi$. Then the inequality (2.1) has a global asymptotics and the attractor A is quasiregular.

6. Stability analysis of OCEVI's on the base of measurements

Consider with $q \in Q$ the parameter-dependent OCEVI

$$\left.\begin{array}{l}(\dot{y} - A(q)y - B(q)\xi\,,\, y - \eta)_{-1,1} + \\ + \psi(\eta) - \psi(y) \geq 0, \forall \eta \in Y_1 \\ y(0) = y_0 \in Y_0 \\ \xi(t) \in \varphi(t, w(t)), \\ w(t) = C(f)y, \\ z(t) = D(q)y + E(q)\xi\,. \end{array}\right\} (6.1)_q$$

Let Q be a metric space with metric d
For any $q \in Q$ we suppose:

$A(q) : \mathcal{D}(A(q)) \to Y_0$ is generator of a C_0-semigroup on Y_0,
$B(q) \in \mathcal{L}(\Xi, Y_{-1}), C(q) \in \mathcal{L}(Y_1, W)$,
$D(q) \in \mathcal{L}(Y_1, Z)$,
$E(q) \in \mathcal{L}(\Xi, Z)$.
For $y \in Q$ and $s \in \mathbb{C} \setminus \rho(A)$ define

$$\left.\begin{array}{rcl} \mathcal{X}^{(w)}(s,q) & = & C(q)\big(sI - A(q)\big)^{-1}B(q) \\ \mathcal{X}^{(z)}(s,q) & = & D(q)\big(sI - A(q)\big)^{-1}B(q) + E(q) \end{array}\right\} \text{transfer operators}$$

Introduce the nonlinearities $\varphi : \mathbb{R}_+ \times W \to 2^{\Xi}$,

with $\varphi \in \mathcal{N}(F(\cdot,\cdot,q))$ were F is given by

$F(w,\xi,q) = (F_1(q)w,w)_W + 2\text{Re}\,(F_2(q)w,\xi)_\Xi + (F_3(q)\xi,\xi)_\Xi$ with

$$F_1(q) = F_1(q)^* \in \mathcal{L}(W), F_2(q) \in \mathcal{L}(W,\Xi),$$

$$F_3(q) = F_3(q)^* \in \mathcal{L}(\Xi)\,.$$

Define by $J_\nu(\cdot,\cdot) : Q \times \mathcal{T} \to \mathbb{R}\,, \nu = 1,2,\dots,k$, stability functionals, where $\mathcal{T}$ is a Hilbert space.
Assume $J = (J_1,\dots,J_k) \in \mathcal{S}$ (a function space),

$$\tilde{Q}(\tau) := \{q \in Q : J_\nu(q,\tau) \le 0, \nu = 1,2,\dots,k\}.$$

Suppose $Q_{abs} \subset Q$ is the set of all $q \in Q$ s.t.$(6.1)_q$ is absolute stable with respect to the observation $z(\cdot)$ in the class $\mathcal{N}(F(\cdot,\cdot,f))$

$$\Leftrightarrow \exists \tau_{abs} \in \mathcal{T} \text{ s.t. } Q_{abs} = \tilde{Q}(\tau_{abs})\,.$$

Consider for $N = 1,2,\dots$ the observation operators D^N and E^N, the observation spaces Z^N and the parameter spaces $\mathcal{T}^{\mathcal{M}}$ s.t.

$$z^N(t) = D^N y(t) + E^N \xi(t) \qquad (6.2)_N$$

with $D^N : Y \to Z^N, E^N : \Xi \to Z^N$,
$Z^N \subset Z, \mathcal{T}^M \subset \mathcal{T}$ finite dimensional subspaces. Assume $\tilde{Q}(\tau^M) = \{q \in Q : J_\nu(q,\tau^M) \le$

$\leq 0, \nu = 1, 2, \ldots, k\}$

and $Q_{abs}(N) \subset Q$ is the set of all $q \in Q$ s.t. $(6.1)_q, (6.2)_N$ is absolutely stable with respect to the observation $z^N(\cdot)$ in the class $\mathcal{N}(F(\cdot, \cdot, q)) \Leftrightarrow$
$\exists\, M\, \exists \tau_{abs}^M \in \mathcal{T}^M$ s.t. $Q_{abs}(N) = \tilde{Q}(\tau^M)$.

Theorem 6.1 Suppose that $\tau_{abs}^M \to \tau$ for

$N \to \infty$ and

$M \to \infty$ in $\mathcal{T}$. Then $\tilde{Q}(\tau) = Q_{abs}$.