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Frequency domain conditions for the existence of almost periodic solutions in evolutionary variational inequalities

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Abstract. Evolutionary variational inequalities are considered as control systems in a general rigged Hilbert space structure. Sufficient frequency domain conditions for boundedness and the existence of periodic and almost periodic solutions are derived. As an example a boundary control problem with periodic exitation function is considered.

Keywords: variational inequality; periodic and almost periodic solutions; frequency domain methods
AMS subject classification: Primary Primary 34G20, 44A05, secondary 47D06

1 Introduction

Frequency domain conditions for the existence of periodic and almost periodic solutions of ODE's are derived in [8, 16, 18, 19]. In [1, 4, 10, 15] sufficient conditions for the existence of almost periodic solutions for evolutionary inequalities and equations are proven using the theory of monotone operators. For many systems for which monotonicity properties are not satisfied it is possible to construct certain Lyapunov functionals as solutions of associated operator inequalities. As a byproduct of such Lyapunov functionals one can derive a priori estimates for the solutions which give some information about their global existence.

We consider in this paper bounded, periodic and almost periodic solutions of a class of evolutionary inequalities or equations the linear part of which is defined by linear operators acting in a rigged Hilbert space structure. Such systems occur naturally if one considers boundary control problems ([7, 9, 14]). Another origin for these systems is the realization theory of dynamical systems. It was shown in [17] that if one has an observation (time-series) with a semigroup property of a well-posed input-output process then there exists a realization of this observation as solution of a dynamical system given in a rigged Hilbert space structure.

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The paper is organized as follows. In Section 2 we describe a class of evolutionary inequalities as generalized feedback-control inequalities. We characterize the "linear part" of such inequalities and introduce certain constraints for the resulting "nonlinearities". In Section 3 frequency-domain conditions for the existence of bounded solutions for the evolutionary inequalities are derived.
Unlike the finite-dimensional case the solution of the operator Lyapunov inequality with a given stable system operator is not necessarily coercitive. In order to get the missing coercitiveness of the Lyapunov functional we add to the quadratic form some form which comes from generalized gradient like properties of the nonlinearity. In Section 4 this Lyapunov functional is used to prove the existence of periodic or almost periodic solutions. In the last Section 5 we discuss the existence of periodic and almost periodic solutions for a boundary control problem connected with the heat equation ([5, 12]).

2 Evolutionary variational inequalities

Let us start with the construction of a rigged Hilbert space structure as it is defined in [3]. Suppose that Y_0 is a real Hilbert space with $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ as scalar product resp. norm. Suppose also that on Y_0 there is an unbounded self-adjoint operator Λ with dense domain $\mathcal{D}(\Lambda)$ such that

$$(\Lambda y, \Lambda y)_0 \geq \|y\|_0^2\,, \quad \forall\, y \in \mathcal{D}(\Lambda)\,.$$

Then $Y_1 := \mathcal{D}(\Lambda)$ is a Hilbert space with the scalar product

$$(y,\eta)_1 := (\Lambda y, \Lambda \eta)_0\,, \quad \forall\, y,\eta \in \mathcal{D}(\Lambda)\,, \tag{2.1}$$

Consider in Y_0 the new scalar product

$$(y,\eta)_{-1} := (\Lambda^{-1} y, \Lambda^{-1}\eta)_0\,, \quad \forall\, y,\eta \in Y_0\,, \tag{2.2}$$

and let Y_{-1} be the completion of Y_0 with respect to this scalar product. It is clear that Y_{-1} is a Hilbert space. We denote the scalar product and norm of Y_{-1} by $(\cdot,\cdot)_{-1}$ and $\|\cdot\|_{-1}$, respectively. Thus we have the dense and continuous embedding $Y_1 \subset Y_0 \subset Y_{-1}$ which is called *rigged Hilbert space structure.* It follows from above that for $y \in Y_1$ and $\eta \in Y_0$ we have

$$|(\eta, y)_0| = |(\Lambda^{-1}\eta, \Lambda y)_0| \leq \|\Lambda^{-1}\eta\|_0\, \|\Lambda y\|_0 = \|\eta\|_{-1}\|y\|_1\,.$$

Extending by continuity the functionals $(\cdot, y)_0$ onto Y_{-1} we obtain the bilinear form $(\cdot,\cdot)_{-1,1}$ ("scalar product") on $Y_{-1}\times Y_1$, coinciding with $(\cdot,\cdot)_0$ on $Y_0 \times Y_1$ and satisfying $|(\eta, y)_{-1,1}| \leq \|\eta\|_{-1}\|y\|_1\,, \quad \forall\, \eta \in Y_{-1}, y \in Y_1$.
If $-\infty \leq T_1 < T_2 \leq +\infty$ are two arbitrary numbers, we define the norm for Bochner measurable functions in $L^2(T_1, T_2; Y_j)$, $j = 1, 0, -1$, by

$$\|y\|_{2,j} := \Big(\int_{T_1}^{T_2} \|y(t)\|_j^2 \, dt \Big)^{1/2}\,. \tag{2.3}$$

Let $\mathcal{W}(T_1,T_2)$ denote the space of functions y such that $y \in L^2(T_1,T_2;Y_1)$ and $\dot{y} \in L^2(T_1,T_2;Y_{-1})$ equipped with the norm

$$\|y\|_{\mathcal{W}(T_1,T_2)} := (\|y\|_{2,1}^2 + \|\dot{y}\|_{2,-1}^2)^{1/2} . \tag{2.4}$$

By an embedding theorem ([13]) one can assume that any function from $\mathcal{W}(T_1,T_2)$ belongs to $C(T_1,T_2;Y_0)$. Assume now that Ξ is a real Hilbert space with scalar product $(\cdot,\cdot)_\Xi$ and norm $\|\cdot\|_\Xi$, respectively, and

$$A : Y_1 \to Y_{-1} \quad \text{and} \quad B : \Xi \to Y_{-1} \tag{2.5}$$

are linear continuous operators. Define also the maps

$$\varphi : Y_1 \to \Xi\,, \tag{2.6}$$

$$\psi : Y_1 \to \mathbb{R}_+\,, \tag{2.7}$$

and

$$f : \mathbb{R} \to Y_{-1} \tag{2.8}$$

and consider for a.a. $t \in [T_1,T_2]$ the evolutionary variational inequality

$$(\dot{y}(t) - Ay(t) - B\varphi(y(t)) - f(t)\,,\; \eta - y(t))_{-1,1} + \psi(\eta) - \psi(y(t)) \geq 0\,, \quad \forall\, \eta \in Y_1\,. \tag{2.9}$$

A function $y(\cdot) \in \mathcal{W}(T_1,T_2) \cap C(T_1,T_2;Y_0)$ is said to be a *solution* of (2.9) if this inequality is satisfied for a.a. $t \in [T_1,T_2]$.

Remark 2.1 a) In applications the operators A and B represent the linear part of the system, φ is a material law nonlinearity, ψ is a contact-type or friction-type functional, and f is an outer perturbation.
b) In the contact-free case when $\psi \equiv 0$ the evolutionary variational inequality (2.5) – (2.9) is equivalent to the evolution equation on Y_{-1}

$$\dot{y}(t) = Ay(t) + B\varphi(y(t)) + f(t)\,. \tag{2.9$'$}$$

c) In practice the linear operators A and B from (2.5) can be constructed as follows ([3]). Suppose that Y_1 is a Hilbert space with scalar product $(\cdot,\cdot)_1$ and norm $\|\cdot\|_1$, respectively. Let $a(\cdot,\cdot)$ be a continuous bilinear form on $Y_1 \times Y_1$. Choose now an arbitrary Hilbert space Y_0 with scalar product $(\cdot,\cdot)_0$ and norm $\|\cdot\|_0$, respectively, such that the embedding $Y_1 \subset Y_0$ is dense and continuous. It is shown in [3] that then there exists a linear unbounded self-adjoint operator Λ with $\mathcal{D}(\Lambda) = Y_1$ and $\|\Lambda y\|_0 = \|y\|_1\,,\ \forall\, y \in Y_1$. Introduce w.r.t. Λ the Hilbert space Y_{-1} as above. Thus we get from $a(\cdot,\cdot)$ the map $A \in \mathcal{L}(Y_1,Y_{-1})$ by $a(y,\eta) = (Ay,\eta)_{-1,1}\,, \forall\, y,\eta \in Y_1$. Let $A^+ \in \mathcal{L}(Y_1,Y_{-1})$ be the operator which is adjoint to A in the following sense: $(A\eta,y)_{-1,1} = (A^+y,\eta)_{-1,1}, y \in Y_1, \eta \in Y_1$. In order to define B we consider on $\Xi \times Y_1$ the continuous bilinear form $b(\cdot,\cdot)$ which corresponds to an operator $B \in \mathcal{L}(\Xi,Y_{-1})$ through $b(\xi,y) = (B\xi,y)_{-1,1}$, $\forall \xi \in \Xi$, $\forall y \in Y_1$. □

Throughout this paper we use the following assumptions about the variational inequality (2.9).

(A1) $\varphi\,(0) = 0\,.$

(A2) There exists operators $N \in \mathcal{L}(Y_1, \Xi)$ and $M = M^* \in \mathcal{L}(\Xi, \Xi)$ such that

$$(\varphi(y_1) - \varphi(y_2), N(y_1, -y_2))_\Xi \geq (\varphi(y_1) - \varphi(y_2), M(\varphi(y_1) - \varphi(y_2)))_\Xi , \quad \forall\, y_1, y_2 \in Y_1 . \tag{2.10}$$

(A3) There exists a quadratic form $\mathcal{G}$ on $Y_1 \times \Xi$ and a continuous functional $\Phi : Y_0 \to \mathbb{R}_+$ such that on any time interval $[T_1, T_2]$, for two arbitrary elements $y_1(\cdot), y_2(\cdot) \in \mathcal{W}(T_1, T_2)$, and a.a. $s, t \in [T_1, T_2], s < t$, we have

$$\int_s^t \mathcal{G}(y_1(\tau) - y_2(\tau), \varphi(y_1(\tau)) - \varphi(y_2(\tau)))d\tau \geq \frac{1}{2}\,\Phi(y_1(\tau) - y_2(\tau))|_s^t . \tag{2.11}$$

Furthermore, there are two constants $0 < \rho_1 < \rho_2$ such that

$$\rho_1 \|y\|_0^2 \leq \Phi(y) \leq \rho_2 \|y\|_0^2 , \quad \forall\, y \in Y_0. \tag{2.12}$$

In addition to **(A1)** – **(A3)** we suppose that there exists a number $\lambda > 0$ such that the following assumptions are satisfied:
(A4) For any $T > 0$ and any $f \in L^2(0, T; Y_{-1})$ the problem

$$\dot{y} = (A + \lambda I)\, y + f(t),\ y(0) = y_0, \tag{2.13}$$

is well-posed, i.e. for arbitrary $y_0 \in Y_0,\ f(\cdot) \in L^2(0, T; Y_{-1})$ there exists a unique solution $y(\cdot) \in \mathcal{W}(0, T)$ satisfying the equation (2.13) in a variational sense and depending continuously on the initial data, i.e.

$$\|y(\cdot)\|^2_{\mathcal{W}(0,T)} \leq c_1 \|y_0\|_0^2 + c_2 \|f(\cdot)\|^2_{2,-1} , \tag{2.14}$$

where $c_1 > 0$ and $c_2 > 0$ are some constants.
(A5) Any solution of $\dot{y} = (A + \lambda I)\, y$, $y(0) = y_0 \in Y_0$ is exponentially decreasing for $t \to +\infty$, i.e. there exist constants $c_3 > 0$ and $\varepsilon > 0$ such that

$$\|y(t)\|_0 \leq c_3\, e^{-\varepsilon t} \|y_0\|_0 ,\ t > 0 . \tag{2.15}$$

(A6) The operator $A + \lambda I \in \mathcal{L}(Y_1, Y_{-1})$ is regular, i.e. for any $T > 0$, $y_0 \in Y_1, z_T \in Y_1$ and $f \in L^2(0, T; Y_{-1})$ the solution of the direct problem

$$\dot{y} = (A + \lambda I)y + f(t),\ y(0) = y_0 ,$$

and the solution of the adjoint problem

$$\dot{z} = -(A + \lambda I)^+ z + f(t),\ z(T) = z_T$$

are strongly continuous in t in the norm of Y_1.
(A7) The pair $(A + \lambda I, B)$ is L^2-controllable, i.e. for arbitrary $y_0 \in Y_0$ there exists a control $\xi(\cdot) \in L^2(0, +\infty; \Xi)$ such that the problem

$$\dot{y} = (A + \lambda I)y + B\xi ,\ y(0) = y_0,$$

is well-posed in the variational sense on $(0,+\infty)$.

Let us denote by H^c and L^c the complexification of a linear space H and a linear operator L, respectively, and introduce by $\chi(s)=(sI_c-A^c)^{-1}B^c$, $s\in\rho(A^c)$, the transfer operator, and by $\mathcal{G}^c$ the Hermitian extension of $\mathcal{G}$. Introduce also for any parameter $\Theta\in\mathbb{R}$ on $Y_1^c\times\Xi^c$ the Hermitian form

$$\mathcal{F}^c(y,\xi;\Theta)=\Theta[\mathrm{Re}(\xi,N^cy)_{\Xi^c}-(\xi,M^c\xi)_{\Xi^c}]+\mathcal{G}^c(y,\xi)+\gamma\lambda\rho_2\|y\|^2_{Y_1^c}\,, \tag{2.16}$$

where the operators N and M are from **(A2)**, the quadratic form $\mathcal{G}$ is from **(A3)**. ρ_2 steams from (2.12) and γ is the embedding constant from $Y_1\subset Y_0$.

(A8) a) For all $\omega\in\mathbb{R}$ there exists the continuous inverse operator $((i\omega-\lambda)I_c-A_c)^{-1}$;
b) There exists a number $\Theta>0$ such that

$$\mathcal{F}^c(\chi(i\omega-\lambda)\xi,\xi;\Theta)<0\,,\quad\forall\xi\in\Xi^c,\xi\neq 0\,. \tag{2.17}$$

In case when the property a) of **(A8)** is not satisfied we use the following assumption.

(A8)' There exists a $\Theta>0$ such that

$$\inf_{(\omega,y,\xi)\in U}-\frac{\mathcal{F}^c(y,\xi;\Theta)}{\|y\|^2_{Y_1^c}+\|\xi\|^2_{\Xi^c}}<0\,, \tag{2.18}$$

where $U=\{(\omega,y,\xi)\in\mathbb{R}\times Y_1^c\times\Xi^c\,|\,i\omega y=A^cy+B^c\xi\}$.

(A9) Suppose that N and M are the operators from **(A2)**, $\mathcal{G}$ is the quadratic form from **(A3)**, Θ is from **(A8)** or **(A8)**' and γ is the embedding constant from $Y_1\subset Y_0$. If there exists a non-negative operator $P=P^*\in\mathcal{L}(Y_0,Y_0)$ such that $P\in\mathcal{L}(Y_{-1},Y_0)\cap\mathcal{L}(Y_0,Y_1)$ and a number $\delta>0$ as solution of the inequality

$$\begin{aligned}((A+\lambda I)\,y+B\,\xi,Py)_{-1,1}+\Theta\big[(\xi,Ny)_\Xi-(\xi,M\xi)_\Xi\big]+\mathcal{G}\,(y,\xi)+\gamma\lambda\,\rho_2\,\|y\|_1^2\\ \leq\;-\delta\left[\|y\|_1^2+\|\xi\|^2_\Xi\right],\quad\forall\,\xi\in\Xi,\,\forall\,y\in Y_1\,,\end{aligned} \tag{2.19}$$

then

$$\begin{aligned}\psi(y_1)-\psi(y_1-P(y_1-y_2))+\psi(y_2)-\psi(y_2+P(y_1-y_2))\geq 0\\ \forall\,y_1,y_2\in Y_1\,.\end{aligned} \tag{2.20}$$

On Y_1 the function $\psi_P(y):=\psi(y-Py)-\psi\,(y)$ is convex and lower continuous.

(A10) a) For any $f\in L^2_{\rm loc}([t_0,+\infty);Y_{-1})$ and any $y_0\in Y_0$ there exists a unique solution y of (2.9) on $[t_0,\infty)$ with $y(t_0)=y_0$. On any finite interval $[t_0,t_0+T]$ this solution satisfies the inequalities

$$\|y\|_{L^2(t_0,t_0+T;Y_1)}\leq c_1(\|f\|_{L^2(t_0,t_0+T;Y_{-1})},\|y_0\|_0) \tag{2.21}$$

and

$$\|y\|_{C(t_0,t_0+T;Y_0)}\leq c_2(\|f\|_{L^2(t_0,t_0+T;Y_{-1})},\|y_0\|_0) \tag{2.22}$$

where $c_i(\cdot,\cdot), i = 1,2$, are continuous non decreasing with respect to each variable functions. The solution y can be approximated in $L^2(t_0, t_0+T; Y_1)$ by a sequence of functions $y_n \in C^1(t_0, t_0+T; Y_1),\ n = 1,2,\dots$.
b) If $y_n \in \mathcal{W}(T_1,T_2) \cap C(T_1,T_2;Y_0), n = 1,2,\dots,$ are solutions of (2.9) with $f = f_n \in L^2(T_1,T_2;Y_{-1})$ and $\lim_{n\to\infty} f_n = \tilde{f}$ in $L^2(T_1,T_2;Y_{-1})$, $\lim_{n\to\infty} y_n = y$ in $C(T_1,T_2;Y_0)$ and weakly in $L^2(T_1,T_2;Y_1)$ then $y \in \mathcal{W}(T_1,T_2) \cap C(T_1,T_2;Y_0)$ is a solution of (2.9) with outer perturbation $f = \tilde{f}$.

Remark 2.2 a) Assumption **(A2)** can be considered as an infinite-dimensional generalization of the sector condition for the nonlinearity φ coming from absolute stability theory ([18]). Assumption **(A3)** means that there is a generalized potential Φ for the nonlinearity φ.
b) Under the assumption **(A4)** the linear system (2.13) with $f \equiv 0$ generates a C_0-semigroup of operators in Y_0. The operator $A + \lambda I : Y_1 \to Y_{-1}$ can be considered as an extension onto the space Y_1 of the generator $A + \lambda I : \mathcal{D}(A + \lambda I) \to Y_0$ of the semigroup.
c) If $B \in \mathcal{L}(\Xi, Y_0)$ the regularity condition **(A6)** is not required. In this case the solution theory for the linear system can be considered in the framework of semigroups, i.e. under weaker conditions than **(A6)**.
d) The assumption **(A8)** (or **(A8)**') is the frequency-domain condition for the solvability of operator inequalities by the Likhtarnikov-Yakubovich frequency theorem ([12]).
e) In assumption **(A9)** a generalized non-negativity condition for the contact-type part is introduced. Unfortunately, for this one has to solve the operator inequality (2.19). Note, however, that for important classes of operators which occur in (2.19) there exist efficient algorithms for finding P which are based on operator symbols ([11]).
f) Part a) from (**A10**) summarizes standard existence and uniqueness properties of solutions of variational inequalities quoted in [4, 15]. Part b) from (**A10**) describes the passage to the limit property in the variational inequality as it is shown for classes of evolutionary inequalities in [15]. □

3 Existence of a bounded solution

In this section we show the existence of at least one bounded on $\mathbb{R}$ solution of (2.9). Together with the exponential stability of solutions, which will be also shown, we derive the uniqueness of such a bounded solution.
Let $J = [T_1, T_2]$ be a bounded or unbounded interval of $\mathbb{R}$ and Y be a real Hilbert space. By $C_b(J;Y) \subset C(J;Y)$ we denote the subspace of all bounded functions equipped with the norm $\|f\|_{C_b} := \sup_{t\in J} \|f(t)\|$. Here $\|\cdot\|$ is the norm generated by the scalar product in Y.
Let $J = \mathbb{R}$ or $J = [T_1, +\infty)$. The space $BS^2(J;Y)$ consists of all functions $f \in L^2_{\mathrm{loc}}(J;Y)$ for which the value

$$\|f\|^2_{S^2} := \sup_{t\in J} \int_t^{t+1} \|f(\tau)\|^2 d\tau$$

is finite.

Lemma 3.1 *Assume that the assumptions* **(A1)** – **(A9)** *are satisfied. Then there exists a positive operator* $P = P^* \in \mathcal{L}(Y_0, Y_0)$ *such that* $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ *and the functional*

$$V(y) := \frac{1}{2}(y, Py)_0 + \frac{1}{2}\Phi(y), \quad y \in Y_0, \tag{3.1}$$

with Φ *from (2.12) has the following properties:*
a) If $y_1(\cdot), y_2(\cdot)$ *are solutions of (2.9) on* $J = [T_1, \infty)$ *with* $f = f_i \in L^2_{\text{loc}}(J; Y_{-1})$, $i = 1, 2$, *then for any* $s, t \in J$, $s \le t$, *we have*

$$\begin{aligned} &V(y_1(\tau) - y_2(\tau))|_s^t + 2\lambda \int_s^t V(y_1(\tau) - y_2(\tau))\, d\tau \\ &\le \int_s^t (f_1(\tau) - f_2(\tau), P(y_1(\tau) - y_2(\tau)))_{-1,1}\, d\tau - \delta \int_s^t \|y_1(\tau) - y_2(\tau)\|_1^2 d\tau. \end{aligned} \tag{3.2}$$

b) If $y_1(\cdot), y_2(\cdot)$ *are two solutions of (2.9) on* $J = [T_1, \infty)$ *with* $f \in L^2_{\text{loc}}(J; Y_{-1})$, *then for any* $t_0 \in J$ *and all* $t \ge t_0$ *we have*

$$V(y_1(t) - y_2(t)) \le e^{-2\lambda(t - t_0)}\, V(y_1(t_0) - y_2(t_0)). \tag{3.3}$$

The number $\lambda > 0$ *in (3.2) and (3.3) comes from* **(A4)** – **(A9)**; *the number* $\delta > 0$ *in (3.2) does not depend on the solutions* y_1, y_2.

Proof Due to the assumptions **(A5)** – **(A9)** from the Likhtarnikov-Yakubovich frequency-theorem ([12]) it follows that there exists an operator $P = P^* \in \mathcal{L}(Y_0, Y_0)$ such that $P \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$, and a number $\delta > 0$ such that

$$\begin{aligned} &((A + \lambda I)y + B\xi,\ Py)_{-1,1} + \Theta\left[(\xi, Ny)_\Xi - (\xi, M\xi)_\Xi\right] + \mathcal{G}(y, \xi) + \gamma\lambda\rho_2\|y\|_1^2 \\ &\qquad \le -\delta\left[\|y\|_1^2 + \|\xi\|_\Xi^2\right], \quad \forall\, y \in Y_1, \forall\, \xi \in \Xi. \end{aligned} \tag{3.4}$$

If we put in (3.4) $\xi = 0$ we get the inequality

$$((A + \lambda I)y,\ Py)_{-1,1} \le -\delta\|y\|_1^2, \quad \forall\, y \in Y_1. \tag{3.5}$$

Using the assumption **(A5)** it follows from (3.5) that $P > 0$. Note that P is not necessarily coercive. In order to get this property we consider the functional (3.1). Due to the property $P > 0$ and the assumption **(A3)** V is coercive.
Now let us prove a). With respect to the solution y_1 we consider for a.a. $t \in J$ the test function $\eta = y_1 + P(y_2 - y_1)$ in order to derive from (2.9) with $f = f_1$ the inequality (we suppress t in y_i)

$$\begin{aligned} &(\dot{y}_1 - Ay_1 - B\varphi(y_1) - f_1(t),\ P(y_2 - y_1))_{-1,1} \\ &\qquad + \psi(y_1 + P(y_2 - y_1)) - \psi(y_1) \ge 0. \end{aligned} \tag{3.6}$$

With respect to the solution y_2 of (2.9) with $f = f_2$ we consider for a.a. $t \in J$ the test function $\eta = y_2 - P(y_2 - y_1)$. This gives

$$\begin{aligned} &(\dot{y}_2 - Ay_2 - B\varphi(y_2) - f_2(t),\ -P(y_2 - y_1))_{-1,1} \\ &+ \psi(y_2 - P(y_2 - y_1)) - \psi(y_2) \ge 0. \end{aligned} \tag{3.7}$$

If we add the inequalities (3.6) and (3.7) we receive

$$
\begin{aligned}
&(\dot y_1 - \dot y_2, P(y_2-y_1))_{-1,1} + (A(y_2-y_1) + B[\varphi(y_2)-\varphi(y_1)] + f_2 - f_1, P(y_2-y_1))_{-1,1}\\
&+ \psi(y_1 + P(y_2-y_1)) - \psi(y_1) + \psi(y_2 - P(y_2-y_1)) - \psi(y_2) \geq 0 \qquad (3.8)
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&(\dot y_2 - \dot y_1, P(y_2-y_1))_{-1,1} - (A(y_2-y_1) + B[\varphi(y_2)-\varphi(y_1)] + f_2 - f_1, P(y_2-y_1))_{-1,1}\\
&+ \psi(y_1) - \psi(y_1 + P(y_2-y_1)) + \psi(y_2) - \psi(y_2 - P(y_2-y_1)) \leq 0\,. \qquad (3.9)
\end{aligned}
$$

From (3.9) and **(A9)** it follows that for a.a. $t \in J$

$$
\begin{aligned}
&(\dot y_2 - \dot y_1,\ P(y_2-y_1))_{-1,1}\\
&- (A(y_2-y_1) + B[\varphi(y_2) - \varphi(y_1)] + f_2 - f_1, P(y_2-y_1))_{-1,1} \leq 0\,. \qquad (3.10)
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
&(\dot y_2 - \dot y_1,\ P(y_2-y_1))_{-1,1} + \lambda(y_2-y_1,\ P(y_2-y_1))_0\\
&- ((A+\lambda I)(y_2-y_1) + B[\varphi(y_2)-\varphi(y_1)] + f_2 - f_1,\ P(y_2-y_1))_{-1,1} \leq 0\,. \qquad (3.11)
\end{aligned}
$$

We use the inequality (3.4) with $y = y_2 - y_1$ and $\xi = \varphi(y_2) - \varphi(y_1)$ to derive from (3.11) the estimate

$$
\begin{aligned}
&(\dot y_2 - \dot y_1,\ P(y_2-y_1))_{-1,1} + \lambda(y_2-y_1, P(y_2-y_1))_0\\
&+ \Theta[(\varphi(y_2)-\varphi(y_1), N(y_2-y_1))_\Xi - (\varphi(y_2)-\varphi(y_1), M(\varphi(y_2)-\varphi(y_1)))_\Xi]\\
&+ \mathcal{G}(y_2-y_1, \varphi(y_2)-\varphi(y_1)) + \gamma\rho_2\lambda\|y_2-y_1\|_1^2 - (f_2-f_1, P(y_2-y_1))_{-1,1}\\
&+ \delta[\|y_2-y_1\|_1^2 + \|\varphi(y_2)-\varphi(y_1)\|_\Xi^2] \leq 0\,. \qquad (3.12)
\end{aligned}
$$

Along the solution pair y_1, y_2 we have according to **(A2)** the property

$$
\Theta[(\varphi(y_2)-\varphi(y_1), N(y_2-y_1))_\Xi - (\varphi(y_2)-\varphi(y_1), M(\varphi(y_2)-\varphi(y_1)))_\Xi] \geq 0\,. \qquad (3.13)
$$

Integration of (3.12) on $[s,t] \subset J$ under consideration of (3.13) gives

$$
\begin{aligned}
&\frac{1}{2}(y_2-y_1,\ P(y_2-y_1))_0\big|_s^t + \lambda\int_s^t (y_2-y_1, P(y_2-y_1))_0\,d\tau\\
&+ \int_s^t (f_1(\tau)-f_2(\tau), P(y_2-y_1))_{-1,1}d\tau + \int_s^t \mathcal{G}(y_2-y_1, \varphi(y_2)-\varphi(y_1))\,d\tau\\
&+ (\gamma\rho_2\lambda + \delta)\int_s^t \|y_2-y_1\|_1^2\,d\tau \leq 0\,. \qquad (3.14)
\end{aligned}
$$

In (3.14) we have used $\frac{1}{2}(y_2-y_1, P(y_2-y_1))_0\big|_s^t = \int_s^t(\dot y_2 - \dot y_1, P(y_2-y_1)_{-1,1}d\tau$, a property which is satisfied for C^1 functions y_1, y_2. According to (**A10**) we can choose two sequences of C^1 functions which approximate y_1 and y_2 in $L^2(J;Y_1)$, integrate the inequality (3.14) with some $\varepsilon_n > 0, \varepsilon_n \to 0$, on the right-hand side and then pass to the limit for $n \to \infty$.

From **(A3)** it follows that

$$\int_s^t \mathcal{G}\left(y_2 - y_1,\, \varphi\left(y_2\right) - \varphi\left(y_1\right)\right) d\tau + \gamma\rho_2\lambda \int_s^t \|y_2 - y_1\|_1^2 \, d\tau$$
$$\geq \frac{1}{2}\,\Phi\left(y_2 - y_1\right)|_s^t + \lambda \int_s^t \Phi\left(y_2 - y_1\right) d\tau\,. \tag{3.15}$$

Using (3.15) we derive from (3.14) the inequality

$$\frac{1}{2}\left[(y_2 - y_1,\; P(y_2 - y_1))_0 + \Phi\left(y_2 - y_1\right)\right]|_s^t$$
$$+ 2\lambda \int_s^t \left[\frac{1}{2}\left(y_2 - y_1,\; P\left(y_2 - y_1\right)\right)_0 + \frac{1}{2}\,\Phi\left(y_2 - y_1\right)\right] d\tau$$
$$\leq \int_s^t (f_2(\tau) - f_1(\tau),\, P(y_2 - y_1))_{-1,1}\, d\tau - \delta \int_s^t \|y_2 - y_1\|_1^2 \, d\tau\,. \tag{3.16}$$

Now we prove b). From (3.16) we conclude that with $f = f_1 = f_2$ the function

$$m(t) := \frac{1}{2}\left[\left(y_2(t) - y_1(t),\; P(y_2(t) - y_1(t))\right)_0 + \Phi\left(y_2(t) - y_1(t)\right)\right]$$

satisfies the inequality $m(\tau)|_s^t + 2\,\lambda \int_s^t m(\tau)\, d\tau \leq 0$, from which (3.3) follows immediately.

■

Lemma 3.2 *Assume that the assumptions* (**A1**) – (**A10**) *are satisfied and* $f \in BS^2([t_0, \infty); Y_{-1})$. *Then for any solution* y *of (2.9) on* $[t_0, \infty)$ *we have*

$$y \in C_b\left([t_0, +\infty; Y_0\right) \cap BS^2([t_0, +\infty); Y_1)\,,$$
$$\|y\|_{C_b\,([t_0, +\infty); Y_0)} \leq c_1\left(\|\, y(t_0)\,\|_1,\; \|f\|_{S^2}\right) \tag{3.17}$$

and

$$\|y\|_{S^2} \leq c_2\left(\|\, y(t_0)\,\|_1,\; \|f\|_{S^2}\right)\,, \tag{3.18}$$

where $c_i(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2,$ *are non-negative functions, increasing in each variable and depending on the constant* $\delta > 0$ *of the inequality (3.3) and the embedding constant of* $Y_1 \subset Y_0$.

Proof Let us use Lemma 3.1 a) with $T_1 = t_0, f_1 = f, f_2 = 0, y_1 = y$ and $y_2 = 0$. Under the consideration of $2\lambda \int_s^t V(y(\tau))\, d\tau \geq 0$ and the estimate

$$\int_s^t (f(\tau), Py(\tau))_{-1,1}\, d\tau \leq \int_s^t \|f(\tau)\|_{-1}\|y(\tau)\|_1\|P\|\, d\tau$$

we derive from (3.2) for $s, t \in J, s \leq t$ the inequality

$$V(y(\tau))|_s^t + \delta \int_s^t \|y(\tau)\|_1^2 d\tau \leq \|P\| \int_s^t \|f(\tau)\|_{-1}\|y(\tau)\|_1 d\tau\,. \tag{3.19}$$

Now, if we put $m^2(\tau) := V(y(\tau)), n(\tau) := \|y(\tau)\|_1$ and $k(\tau) := \|f(\tau)\|_{-1}$ we can write (3.19) in the form

$$m^2(\tau)|_s^t + \delta \int_s^t n^2(\tau)\, d\tau \le \|P\| \int_s^t n(\tau)k(\tau)\, d\tau \; . \tag{3.20}$$

This inequality was considered in [15], Ch. 2. Note that from $P = P^* \in \mathcal{L}(Y_0, Y_0)$ and (2.12) it follows that for the considered τ

$$m^2(\tau) \le \frac{1}{2}\Big(\|P\|\, \|y(\tau)\|_0^2 + \gamma_2 \|y(\tau)\|_0^2\Big) \le \frac{1}{2}\Big(\|P\| + \gamma_2\Big)\gamma^2 \|y(\tau)\|_1^2 \, , \tag{3.21}$$

where γ is the embedding constant of the continuous embedding $Y_1 \subset Y_0$. From (3.20) and (3.21) it follows, as it was shown in [15], that if $t - s \le 3$, then there exists a $c > 0$ (depending on δ and $\|P\|$) such that for $\tau \ge t_0$

$$m^2(\tau) \le c \max\, \Big[\|k\|_{S^2}^2 \,\big(1 + (\|P\| + \gamma_2)\,\big)\frac{\gamma^2}{2} \, , \|m^2\|_{L^\infty(t_0,t_0+2)}\,\Big] \tag{3.22}$$

and

$$\|n\|_{S^2} \le c \max\, \Big[\|k\|_{S^2} \,\big(1 + (\|P\| + \gamma_2)\,\big)\frac{\gamma^2}{2} \, , \|m^2\|_{L^\infty(t_0,t_0+2)}\Big] \; . \tag{3.23}$$

The assumption (**A10**) (inequality (2.22) with $T = 2$) implies that

$$\|y\|_{L^\infty(t_0,t_0+2;Y_0)} \le c_2\big(\, \|f\|_{L^2(t_0,t_0+2;Y_{-1})}, \|y\|_0\,\big) \, , \tag{3.24}$$

where the function $c_2(\cdot,\cdot)$ is described in (**A10**). Using now the estimate

$$\|f\|_{L^2(t_0,t_0+2;Y_{-1})} \; \le 2\, \|f\|_{S^2} \, , \tag{3.25}$$

which immediately follows from the definition of $\|\cdot\|_{S^2}$, and the inequality (3.22), the claimed result follows from the estimates (3.24) and (3.25). ■

In the next theorem we prove the existence of a unique bounded solution of (2.9) on $\mathbb{R}$. The essential part of the proof in analogous to the one of Theorem 2.5 in [15]. In our situation, however, other assumptions and auxilary results are exploited. For this reason we give the complete proof below.

Theorem 3.1 *Assume that the assumptions* (**A1**) *–* (**A10**) *are satisfied. Then for any* $f \in BS^2(\mathbb{R}; Y_{-1})$ *there exists an unique solution* y *of (2.9) such that* $y \in C_b(\mathbb{R}; Y_0) \cap BS^2(\mathbb{R}; Y_1)$.

Proof The uniqueness of a bounded on $\mathbb{R}$ solution follows from Lemma 3.1, b).
To prove the existence we consider as in [15] the following approximation problem: Find for any natural n a solution y_n of (2.9) such that $y_n(-n) = 0$ and $y_n \in \mathcal{W}([-n, +\infty)) \cap C([-n, +\infty); Y_0)$. It follows from (**A10**) that any such solution exists and is uniquely defined. Put now for any natural n

$$f_n(t) = \begin{cases} f(t), & t \ge -n \, , \\ 0, & t < -n \, , \end{cases}$$

and extend y_n by zero to the whole axis.
By Lemma 3.2 we have

$$\|y_n\|_{C_b(\mathbb{R};Y_0)} \le c \tag{3.26}$$

and

$$\|y_n\|_{S^2} \le c\,, \tag{3.27}$$

where $c > 0$ does not depend on n.
Since $f_n(t) = f_m(t) = f(t)$ for $t \ge t_{n,m} := -\min\{n,m\}$ the inequality (3.26) and Lemma 3.1 b) with $t_0 = t_{n,m}$ imply for $t \ge t_{n,m}$ the estimate

$$V(y_n(t) - y_m(t)) \le e^{-2\lambda(t-t_{n,m})} V(y_{\max\{n,m\}}(t_{n,m}))\,. \tag{3.28}$$

From (3.26) and (2.12) we conclude the existence of a constant $c_1 > 0$, which does not depend on m and n, such that

$$V(y_{\max\{n,m\}}(t_{n,m})) \le c_1\,. \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$V(y_n(t) - y_m(t)) \le c_1 e^{-2\lambda(t-t_{n,m})}\,. \tag{3.30}$$

As $t_{n,m} \to -\infty$ we conclude from (3.30) that $\{y_n\}$ is a Cauchy sequence in $C_b(\mathbb{R};Y_0)$ and consequently in $C(\mathbb{R};Y_0)$. Thus in this space there exists $\lim_{n\to\infty} y_n =: y$. By (3.27) the sequence $\{y_n\}$ is weakly precompact in $L^2_{\text{loc}}(\mathbb{R};Y_1)$. So $\lim\limits_{n\to\infty} y_n = y$ in $L^2_{\text{loc}}(\mathbb{R};Y_0)$. From assumption (**A10**), b) it follows that y is a solution of (2.9).
Passing to the limit in (3.26) and (3.27) we see that $y \in C_b(\mathbb{R};Y_0) \cap BS^2(\mathbb{R};Y_1)$.

■

4 Existence of periodic and almost periodic solutions

In this section we derive frequency-domain conditions for the existence of a periodic or almost periodic solution for the variational inequality (2.9). Note that this result is not a direct generalization of the ODE situation considered in [18] since some properties (coercitiveness) of the solutions of operator inequalities in the infinite-dimensional case are not satisfied.
Let $(E, \|\cdot\|_E)$ be a Banach space and let $f : \mathbb{R} \to E$ be continuous. If $\varepsilon > 0$ is a given number, then $T \in \mathbb{R}$ is called *ε-almost period* of f if $\sup\limits_{t\in\mathbb{R}} \|f(t+T) - f(t)\|_E \le \varepsilon$. The function f is called *Bohr almost periodic* or shortly *almost periodic* if for every $\varepsilon > 0$ there is an $R > 0$ such that every interval $(r, r+R) \subset \mathbb{R}$ $(r \in \mathbb{R})$ contains at least one ε-almost period of f.
In order to guarantee the existence of an almost periodic solution for (2.9) we need some additional assumptions. Other types of such assumptions are connected with the

continuous dependence on parameters of (2.9), see [15]. Note that in the finite-dimensional setting this type of assumptions is not necessary since in this case $Y_1 = Y_0$.

(**A11**) If $f \in L^2_{\rm loc}(\mathbb{R}; Y_{-1})$ is a given perturbation function in (2.9) under the conditions (**A1**) – (**A10**), $\{f(\cdot + \sigma) \,|\, \sigma \in \mathbb{R}\}$ is the set of all translates of f and $\{y_\sigma(\cdot) \,|\, \sigma \in \mathbb{R}\}$ is the set of associated bounded on $\mathbb{R}$ solutions of inequality (2.9) with $y_\sigma \in C_b(\mathbb{R}; Y_0) \cap BS^2(\mathbb{R}; Y_1)$ then there is a constant $\tilde{c} > 0$ such that for arbitrary $\sigma \in \mathbb{R}$

$$\sup_{t \in \mathbb{R}} \|y_\sigma(t)\|_1 \le \tilde{c} \,. \tag{4.1}$$

Theorem 4.1 *Under the assumptions (***A1***) – (***A10***) there exists a unique bounded on $\mathbb{R}$ solution y_* of (2.9). This solution is exponentially stable in the whole, i.e. there exist positive constants $c > 0$ and $\varepsilon > 0$ such that for any other solution y of (2.9) on $[t_0, \infty)$ and any $t \ge t_0$ we have*

$$\|y(t) - y_*(t)\|_0 \le c\, e^{-\varepsilon(t-t_0)} \|y(t_0) - y_*(t_0)\|_0 \,. \tag{4.2}$$

*If f is T-periodic the solution y_* is also T-periodic. If f is almost periodic and (***A11***) is satisfied then y_* is an almost periodic solution.*

Proof The existence and uniqueness of a bounded on $\mathbb{R}$ solution $y_*(\cdot)$ of (2.9) follows from Theorem 3.1. The exponential stability of $y_*(\cdot)$ results from (3.3). If f is T-periodic then $y_*(t+T)$ is also a bounded on $(-\infty, \infty)$ solution of (2.9). Since $y_*(t)$ is the unique bounded solution it follows that $y_*(t) = y_*(t+T)\ \forall\, t \in \mathbb{R}$.
Suppose that f is almost periodic and consider an arbitrary ε-almost period T of f, i.e.

$$\sup_{t \in \mathbb{R}} \|f(t+T) - f(t)\|_{-1} \le \varepsilon \,. \tag{4.3}$$

Define the function $w(t) := y_*(t+T) - y_*(t)$ and consider $V(w(t))$. It follows from (3.2) that for an arbitrary interval $[t_0, t]$ we have

$$\begin{aligned} & V(w(\tau))|_{t_0}^{t} + 2\lambda \int_{t_0}^{t} V(w(\tau))\, d\tau \\ \le\ & \int_{t_0}^{t} [-\delta \|w(\tau)\|_1^2 + (f(\tau+T) - f(\tau), Pw(\tau))_{-1,1}]\, d\tau \,. \end{aligned} \tag{4.4}$$

Since P is a bounded operator we have the estimate

$$|(f(\tau+T) - f(\tau), Pw(\tau))_{-1,1}| \le \|f(\tau+T) - f(\tau)\|_{-1}\, \|P\|\, \|w(\tau)\|_1 \,. \tag{4.5}$$

From (4.1) and (4.4) it follows that for a.a. $\tau \in \mathbb{R}$

$$|(f(\tau+T) - f(\tau), Pw(\tau))_{-1,1}| \le 2\, \|P\| \tilde{c} \varepsilon \,. \tag{4.6}$$

By the continuous embedding $Y_1 \subset Y_0$ we have

$$\frac{\delta}{\gamma}\, \|w(\tau)\|_0^2 \le \delta\, \|w(\tau)\|_1^2 \,. \tag{4.7}$$

From (4.4), (4.6) and (4.7) it follows now that on $[t_0, t]$

$$V(w(t)) \leq e^{-\tilde{\lambda}(t-t_0)}\, V\left(w(t_0)\right) + \frac{2\|P\|\tilde{c}\,\varepsilon}{\tilde{\lambda}} \left[1 - e^{-\tilde{\lambda}(t-t_0)}\right] , \tag{4.8}$$

where $\tilde{\lambda} := 2\,\lambda + \frac{\delta}{\gamma^2 \gamma_2}$. Since w exists on $\mathbb{R}$ and $V(w(t_0))$ in (4.8) is bounded we can, for fixed t, choose $t_0 \to -\infty$. It follows that on $\mathbb{R}$

$$V(w(t)) \leq \frac{2\|P\|\tilde{c}\,\varepsilon}{\tilde{\lambda}} \, . \tag{4.9}$$

But the inequality (4.9) shows that any ε-period T of f is also a $\frac{2\|P\|\tilde{c}\,\varepsilon}{\tilde{\lambda}}$-period of y_*, considered in the metric introduced on Y_0 by V. Consequently, the function y_* is almost periodic in Y_0. ■

5 Example

Let us consider the spaces $Y_0 = L^2(0,1)$ and $Y_1 = W^{1,2}(0,1)$, where Y_1 is equipped with the scalar product

$$(u,v)_1 = \int_0^1 (uv + u_x v_x)\, dx \, , \quad \forall\, u,v \in W^{1,2}(0,1) \, . \tag{5.1}$$

Then there exists ([3]) an unbounded self-adjoint in Y_0 operator Λ such that $\mathcal{D}(\Lambda) = Y_1$ and $\|\Lambda y\|_0 = \|y\|_1$, $\forall y \in Y_1$. From the operator Λ we can construct the space Y_{-1} which is the completion of Y_0 under the norm $\|y\|_{-1} = \|\Lambda^{-1} y\|_0$. Thus we get the rigged Hilbert space structure $Y_1 \subset Y_0 \subset Y_{-1}$. Let the operator $A \in \mathcal{L}(Y_1, Y_{-1})$ be introduced by the continuous quadratic form

$$a(u,v) = -\int_0^1 (\alpha u_x v_x + \beta uv) dx \, , \quad \forall\, u,v \in W^{1,2}(0,1) \, , \tag{5.2}$$

where $\alpha > 0$ and $\beta > 0$ are some parameters. Note that by the continuity of a on $Y_1 \times Y_1$ this is possible. We choose the space $\Xi = \mathbb{R}$ and define the linear operator $B : \Xi \to Y_{-1}$ by the form

$$b(\xi, v) = \alpha\, \xi v(1) \, , \quad \forall\, \xi \in \mathbb{R}, \quad \forall\, v \in W^{1,2}(0,1) \, . \tag{5.3}$$

As perturbation f we take a linear continuous functional on $W^{1,2}(0,1)$ given by

$$f(t)[v] = \int_0^1 f_1(x,t)\, v(x) dx \, , \quad \forall\, v \in W^{1,2}(0,1) \, , \tag{5.4}$$

where $f_1 \in L^2((0,1) \times \mathbb{R})$ is a continuous periodic or almost periodic in t function. According to the embedding theory for $W^{1,2}(0,1) \subset L^2(0,1)$ this definition is meaningful. Let the linear continuous operator $C : L^2(0,1) \to \mathbb{R}$ be defined by

$$Cu = \int_0^1 c(x)\, u(x) dx \, , \quad \forall\, u \in L^2(0,1) \, , \tag{5.5}$$

where $c(\cdot) \in L^2(0,1)$ is a given function. Assume further that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Using the function g we can introduce the nonlinear map $\varphi : L^2(0,1) \to \mathbb{R}$ by

$$u \in L^2(0,1) \mapsto w(\cdot) = Cu(\cdot) \mapsto g(w(\cdot)) \in \mathbb{R} . \tag{5.6}$$

The operators A, B and C and the nonlinearity φ can be considered as linear part and nonlinearity, respectively, of the abstract evolutionary equation (2.10). If the data are smooth this equation can be written as boundary control problem ([5, 12])

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \alpha u_{xx} - \beta u + f_1(x,t) \; , \; 0 < x < 1 \; , \\ u(x,0) &= u_0(x) \in L^2(0,1) \; , \\ u_x(0,t) &= 0 \; , \; u_x(1,t) = g\left(w(t)\right) \; , \\ w(t) &= \int_0^1 c(x)\, u(x,t)\, dx \; . \end{aligned}\right\} \tag{5.7}$$

In order to satisfy for the nonlinearity φ the assumptions (**A2**) and (**A3**) we assume for g the following: There exist constants $\mu_0 > 0$ and $c_1 > 0$ such that

$$0 \le (g(w_1) - g(w_2))(w_1 - w_2) \le \mu_0 (w_1 - w_2)^2 \; , \quad \forall\, w_1, w_2 \in \mathbb{R} \; , \tag{5.8}$$

and on an arbitrary interval $[T_1, T_2]$ we have

$$\int_s^t [CA(y_1 - y_2) + CB(g(Cy_1) - g(Cy_2))][g(Cy_1) - g(Cy_2)]d\tau \ge \frac{c_1}{2} \|Cy_1 - Cy_2\|_0^2 \Big|_s^t \; ,$$
$$\forall\, y_1, y_2 \in \mathcal{W}(T_1, T_2) \; , \text{ a.a. } s, t \in [T_1, T_2], s < t \; . \tag{5.9}$$

From (5.8) it follows that with $\varphi(y) = g(Cy), N = C$ and $M = \frac{1}{\mu_0}[1]$ the assumption (**A2**) is satisfied, i.e.

$$(\varphi(y_1) - \varphi(y_2))(Cy_1 - Cy_2) \ge \frac{1}{\mu_0} (\varphi(y_1) - \varphi(y_2))^2 \; , \quad \forall\, y_1, y_2 \in Y_1 \; . \tag{5.10}$$

The quadratic form $\mathcal{G}$ from (**A3**) is defined through (5.9).
Because of (5.2) we have an $\varepsilon > 0$ such that

$$(Au, u)_{-1,1} \le -\, \alpha \, \|\tfrac{\partial u}{\partial x}\|_0^2 - \beta \, \|u\|_0^2 \le -\, \varepsilon \|u\|_1^2 - \beta \|u\|_0^2 \; , \quad \forall\, u \in W^{1,2}(0,1) \; . \tag{5.11}$$

It follows from [13] that (5.11) implies (**A4**), (**A5**) and (**A7**). The validity of (**A6**) under our conditions is shown in [5, 12].
Let us assume that the considered class of nonlinearities (5.6), satisfying (5.8), (5.9), is so that the existence and uniqueness of solutions and the continuous dependence on parameters in the sense of (**A10**) are given (see, for example, [15]).
It remains to verify the frequency-domain condition (**A8**). Using the (formal) Laplace transform technique we can show that the transfer operator χ for the linear part of (5.7) can be written as

$$\chi(s) = C\tilde{u}(\cdot, s), \;\; s \in \mathbb{C}, \tag{5.12}$$

where $\tilde{u} = \tilde{u}(\cdot, s)$ is the solution of the boundary problem

$$\left.\begin{array}{l} s\tilde{u} = \alpha\,\tilde{u}_{xx} - \beta\,\tilde{u}\,, \quad s \in \mathbb{C} \\ \tilde{u}_x(0,t) = 0,\ \tilde{u}_x(1,t) = 1\,. \end{array}\right\} \tag{5.13}$$

Let $\lambda > 0$ (sufficiently small) and $\Theta > 0$ be positive parameters. Then the frequency-domain condition (**A8**) with the parameter $\mu_0 > 0$ from (5.8) and 1 as embedding constant of $W^{1,2}(0,1) \subset L^2(0,1)$ has with respect to the quadratic constraints (5.9) and (5.10) the form

$$\Theta\,\Big[\mathrm{Re}\,\chi(i\omega - \lambda) - \frac{1}{\mu_0}\Big] + \mathrm{Re}\,i\omega\chi(i\omega - \lambda) + \ \lambda\,|\chi(i\omega - \lambda)|^2 < 0\,, \quad \forall\,\omega \in \mathbb{R}\,. \tag{5.14}$$

Under the above assumptions and the condition (5.14) the Likhtarnikov-Yakubovich frequency theorem ([12]) states the existence of a positive operator $P = P^* \in \mathcal{L}(Y_0, Y_0)$ such that the Lyapunov-type function V used in the theory of Sections 2 – 4 is given (with c_1 from (5.9)) by

$$V(u) = \frac{1}{2}\int_0^1 (Pu)(x)u(x)\,dx + \frac{c_1}{2}\|u\|_0^2\,, \quad \forall\,u \in L^2(0,1)\,. \tag{5.15}$$

If f_1 in (5.7) is a T-periodic function in time the existence of an unique T-periodic solution $y_* \in C_b(\mathbb{R}; Y_0) \cap BS^2(\mathbb{R}; Y_1)$ for the abstract to (5.6) problem (2.10) with $\psi \equiv 0$ follows now from Theorem 4.1.

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