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Observation stability for controlled evolutionary variational inequalities

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Abstract. We derive absolute observation stability and instability results for controlled evolutionary inequalities which are based on frequency-domain characteristics of the linear part of the inequalities. The uncertainty parts of the inequalities (nonlinearities which represent external forces and constitutive laws) are described by certain local and integral quadratic constraints. Other terms in the considered evolutionary inequalities represent contact-type properties of a mechanical system with dry friction. The absolute stability criteria with respect to a class of observation operators (or measurement operators) give the opportunity to prove the weak convergence of arbitrary solutions of inequalities to their stationary sets.

Keywords: Absolute observation stability, evolutionary variational inequalities, frequency-domain conditions
AMS subject classification: Primary 45M05, 45M10, secondary 93B52, 93C10, 93C25

1 Basic notation

Suppose that Y_0 is a real Hilbert space. We denote by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ the scalar product resp. the norm on Y_0. Let $A:\mathcal{D}(A)\to Y_0$ be the generator of a C_0-semigroup on Y_0 and define the set $Y_1:=\mathcal{D}(A)$. Here $\mathcal{D}(A)$ is the domain of A, which is dense in Y_0 since A is a generator. We denote with $\rho(A)$ the resolvent set of A. The spectrum of A, which is the complement of $\rho(A)$, is denoted by $\sigma(A)$. If we define with an arbitrary but fixed $\beta\in\rho(A)\cap\mathbb{R}$ for any $y,\eta\in Y_1$, the value

$$(y,\eta)_1:=((\beta I-A)y,\,(\beta I-A)\eta)_0\ , \tag{1.1}$$

then the set Y_1 equipped with this scalar product $(\cdot,\cdot)_1$ and the corresponding norm $\|\cdot\|_1$ becomes a Hilbert space (different numbers β give different but equivalent norms). Denote by Y_{-1} the Hilbert space which is the completion of Y_0 with respect to the norm $\|y\|_{-1}:=\|(\beta I-A)^{-1}y\|_0$ and which has the corresponding scalar product

$$(y,\eta)_{-1}:=\left((\beta I-A)^{-1}y,\,(\beta I-A)^{-1}\eta\right)_0,\quad \forall y,\eta\in Y_{-1}. \tag{1.2}$$

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Thus, we get the inclusions $Y_1 \subset Y_0 \subset Y_{-1}$, which are dense with continuous embedding, i.e., $Y_\alpha \subset Y_{\alpha-1}\,, \alpha = 1, 0$, is dense and $\|y\|_{\alpha-1} \le c\|y\|_\alpha\,,\ \forall y \in Y_\alpha$. Sometimes ([3, 4])the introduced triple of spaces (Y_1, Y_0, Y_{-1}) is called a *Gelfand triple* . The pair (Y_1, Y_{-1}) is also called *Hilbert rigging* of the *pivot space* Y_0, Y_1 is an *interpolation space* of Y_0, and Y_{-1} is an *extrapolation space* of Y_0 ([7, 33]). Since for any $y \in Y_0$ and $z \in Y_1$ we have

$$|(y, z)_0| = |(\beta I - A)^{-1}y, ((\beta I - A)z)_0| \le \|y\|_{-1}\|z\|_1\,, \tag{1.3}$$

we can extend $(\cdot, z)_0$ by continuity onto Y_{-1} obtaining the inequality

$$|(y, z)_0| \le \|y\|_{-1}\|z\|_1\,, \quad \forall y \in Y_{-1}, \forall z \in Y_1.$$

Let us denote this extension also by $(\cdot,\cdot)_{-1,1}$ and call it *duality product* on $Y_{-1} \times Y_1$. The operator A has a unique extension to an operator in $\mathcal{L}(Y_0, Y_{-1})$ which we denote by the same symbol. Suppose now that $T > 0$ is arbitrary and define the norm for Bochner measurable functions in $L^2(0, T; Y_j)\quad (j = 1, 0, -1)$ through

$$\|y(\cdot)\|_{2,j} := \left(\int_0^T \|y(t)\|_j^2\,dt\right)^{1/2}. \tag{1.4}$$

Let $\mathcal{L}_T$ be the space of functions such that $y \in L^2(0, T; Y_1)$ and $\dot{y} \in L^2(0, T; Y_{-1})$, where the time derivative $\dot{y}$ is understood in the sense of distributions with values in a Hilbert space. The space $\mathcal{L}_T$ equipped with the norm

$$\|y\|_{\mathcal{L}_T} := \left(\|y(\cdot)\|_{2,1}^2 + \|\dot{y}(\cdot)\|_{2,-1}^2\right)^{1/2} \tag{1.5}$$

is a Hilbert space and will be used for the description of solutions to evolutionary systems.

Remark 1.1 a) Let us denote by $C_T := C(0, T; Y_0)$ the Banach space of continuous mappings $y : [0, T] \to Y_0$ provided with the norm

$$\|y(\cdot)\|_{C_T} = \sup_{t\in[0,T]} \|y(t)\|_0\,.$$

It is well known ([7, 26, 33]) that $\mathcal{L}_T$ can be continuously embedded into the space C_T, i.e., every function from $\mathcal{L}_T$, properly altered by some set of measure zero, is a continuous function $y : [0, T] \to Y_0$ and $\|y(\cdot)\|_{C_T} \le \text{const}\cdot\|y(\cdot)\|_{\mathcal{L}_T}$. This embedding property shows that the Cauchy problem (2.1), (2.2) in the next section can be considered.

b) Since the infinite-dimensional evolutionary problems which are investigated in this paper with the help of observation operators are assumed to have uncertain parts, the choice of physically motivated phase spaces, control spaces, and observation spaces (see Sect. 2) is crucial for the measurement process. It is easy to show that in some norms the observation operator can be stable, whereas it may be unstable in other norms. In many cases the described Gelfand triple of spaces is not sufficient for this and we need a *Hilbert scale* $\{Y_\alpha\}_{\alpha\in\mathbb{R}}$ of spaces. For arbitrary $\alpha \ge 0$ we define (with A and β as in (1.1) and (1.2)) the space $Y_\alpha := \mathcal{D}((\beta I - A)^\alpha)$ and the scalar product in Y_α

$$(y, \eta)_\alpha := ((\beta I - A)^\alpha y\,,\ (\beta I - A)^\alpha \eta)_0, \quad \forall\, y, \eta \in Y_\alpha\,. \tag{1.6}$$

When $\alpha < 0$, the scalar product is also introduced by formula (1.6) and the space Y_α is, as Y_{-1}, obtained by the completion of Y_0 with respect to their norm. Furthermore, for any $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha \in (\beta, \gamma)$ the pair (Y_γ, Y_β) is a Hilbert rigging of Y_α ([7]). □

2 Evolutionary variational inequalities

Suppose that $T > 0$ is arbitrary and consider for a.a. $t \in [0, T]$ the observed and controlled evolutionary variational inequality

$$(\dot{y} - Ay - B\xi - f(t), \eta - y)_{-1,1} + \psi(\eta) - \psi(y) \geq 0\,, \qquad \forall \eta \in Y_1 \tag{2.1}$$
$$y(0) = y_0 \in Y_0\,,$$
$$w(t) = Cy(t)\,, \quad \xi(t) \in \varphi(t, w(t))\,, \quad \xi(0) = \xi_0 \in \mathcal{E}(y_0)\,, \tag{2.2}$$
$$z(t) = Dy(t) + E\,\xi(t)\,. \tag{2.3}$$

In (2.1) $-$ (2.3) it is supposed that $C \in \mathcal{L}(Y_{-1}, W), D \in \mathcal{L}(Y_1, Z)$ and $E \in \mathcal{L}(\Xi, Z)$ are linear operators, Ξ, W and Z are real Hilbert spaces, $Y_1 \subset Y_0 \subset Y_{-1}$ is a real Gelfand triple and $A \in \mathcal{L}(Y_0, Y_{-1})$, $B \in \mathcal{L}(\Xi, Y_{-1}), \varphi : \mathbb{R}_+ \times W \to 2^{\Xi}$ is a set-valued map, $\psi : Y_1 \to \mathbb{R}_+$ and $f : \mathbb{R}_+ \to Y_{-1}$ are given nonlinear maps. The calculation of $\xi(t)$ in (2.2) shows that this value in general also depends on certain "initial state" ξ_0 of φ taken from a set $\mathcal{E}(y_0) \subset \Xi$. This situation is typical for hysteresis nonlinearities. We call B *control operator*, C *output operator*, D and E *observation operators*, Ξ *control space*, W *output space*, Z *observation space*, φ (*material law*) *nonlinearity*, ψ (*contact*) *functional* and f *forcing function* (see Fig. 1).

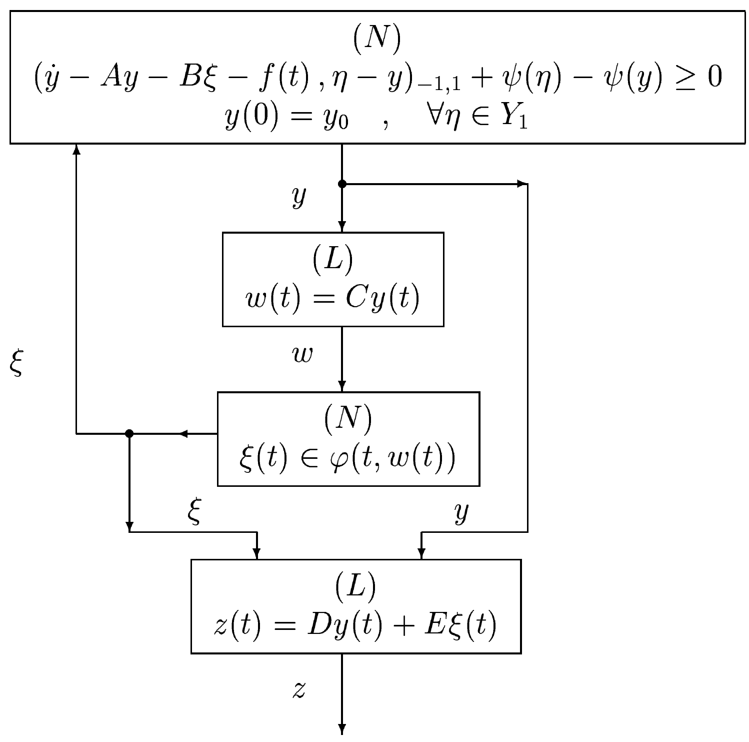

Figure 1: State / linear output / nonlinear output / observation diagram

In the following we denote by $\|\cdot\|_\Xi$, $\|\cdot\|_W$ and $\|\cdot\|_Z$ the norm in Ξ, W resp. Z.

Let us now introduce the solution space for the problem (2.1), (2.2) .

Definition 2.1 *Any pair of functions $\{y(\cdot), \xi(\cdot)\}$ with $y \in \mathcal{L}_T$ and $\xi \in L^2_{\text{loc}}(0,\infty;\Xi)$ such that $B\xi \in \mathcal{L}_T$, satisfying (2.1), (2.2) almost everywhere on $(0,T)$, is called* ***solution of the Cauchy problem*** $y(0) = y_0,\ \xi(0) = \xi_0$ *defined for (2.1), (2.2) .*

In order to have an existence property for $(2.1) - (2.3)$ we state the following assumption:

(C1) The Cauchy-problem (2.1), (2.2) has for arbitrary $y_0 \in Y_0$ and $\xi_0 \in \mathcal{E}(y_0) \subset \Xi$ at least one solution $\{y(\cdot), \xi(\cdot)\}$.

Assumption **(C1)** is fulfilled, for example, in the following situation ([28]).

(C2) a) The nonlinearity $\varphi : \mathbb{R}_+ \times W \to \Xi$ is a function having the property that $\mathcal{A}(t) := -A - B\varphi(t, C\cdot) : Y_1 \to Y_{-1}$ is a family of monotone hemicontinuous operators such that the inequality

$$\|\mathcal{A}(t)y\|_{-1} \leq c_1\|y\|_1 + c_2\,, \quad \forall y \in Y_1\,,$$

is satisfied, where $c_1 > 0$ and $c_2 \in \mathbb{R}$ are constants not depending on $t \in [0,T]$. Furthermore for any $y \in Y_1$ and for any bounded set $U \subset Y_1$ the family of functions $\{(\mathcal{A}(t)\eta, y)_{-1,1}\,,\ \eta \in U\}$ is equicontinuous with respect to t on any compact subinterval of $\mathbb{R}_+$.

b) ψ is a proper, convex, and semicontinuous from below function on $\mathcal{D}(\psi) \subset Y_1$.

(C3) $f \in L^2_{\text{loc}}(\mathbb{R}_+; Y_{-1})$.

Under the assumptions **(C2)** $-$ **(C3)** the following theorem of H. Brézis ([5]) holds:

Theorem 2.1 *Suppose that for the family of operators $\mathcal{A}(t), t \in \mathbb{R}_+$, we have*

$$(\mathcal{A}(t)y, y)_{-1,1} \geq \alpha\|y\|_1^2 + \beta\,, \quad \forall y \in Y_1\,,$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ do not depend on t. Then for arbitrary $f \in L^2_{\text{loc}}(\mathbb{R}_+; Y_{-1})$ and arbitrary $y_0 \in \overline{\mathcal{D}(\psi)}^{Y_0}$ (i.e., the closure in Y_0) there exists a unique weak solution $y \in L^2_{\text{loc}}(\mathbb{R}_+; Y_1) \cap C(\mathbb{R}_+; Y_0)$ with $y(0) = y_0$, satisfying the inequality

$$\int_s^t \Big[\big(\dot\eta(\tau) - \mathcal{A}(\tau)y(\tau) - f(\tau),\ \eta(\tau) - y(\tau)\big)_{-1,1} + \psi(\eta(\tau)) - \psi(y(\tau))\Big]\, d\tau \geq$$
$$\frac{1}{2}\|\eta(t) - y(t)\|_1^2 - \frac{1}{2}\|\eta(s) - y(s)\|_1^2\,,$$
$$\forall s,t:\ 0 \leq s \leq t\,, \qquad \forall \eta \in \big\{\eta \in L^2_{\text{loc}}(\mathbb{R}_+; Y_1) : \dot\eta \in L^2_{\text{loc}}(\mathbb{R}_+; Y_{-1})\big\}\,.$$

Furthermore, the solution satisfies the inequalities

$$\|y\|_{L^2(0,T;Y_1)} \leq c_1 \left(\|f\|_{L^2(0,T;Y_{-1})}\,,\, \|y_0\|_0\right)\,,$$
$$\|y\|_{C(0,T;Y_0)} \leq c_2 \left(\|f\|_{L^2(0,T;Y_{-1})}, \|y_0\|_0\right)\,,$$

where $c_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, *are continuous and monotonically increasing functions.*

(C4) In the sequel we consider only solutions y of (2.1),(2.2) for which $\dot{y}$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}; Y_{-1})$.

Remark 2.1 a) Note that in the special case when $\psi \equiv 0$ in (2.1) the evolutionary variational inequality is equivalent for a.a. $t \in [0, T]$ to the equation

$$\begin{aligned}
&\dot{y} = Ay + B\xi + f(t) \quad \text{in } Y_{-1}\,,\\
&y(0) = Y_0\,,\ w(t) = Cy(t)\,,\ \xi(t) \in \varphi(t, w(t))\,,\ \xi(0) \in \mathcal{E}(y_0)\,,\\
&z(t) = Dy(t) + E\xi(t).
\end{aligned}$$

Under the assumption that φ is a single valued function this class was considered in [3, 4, 6, 7, 8, 10, 18, 21, 31, 36].

b) An important class of uncertainty systems (2.1), (2.2) are connected with hysteresis operators. Let us describe one example of this type which in more details is discussed in [35]. Let us assume that γ_e and γ_r are maximal monotone multivalued functions $\mathbb{R} \to 2^{\bar{\mathbb{R}}}$, such that $\inf \gamma_r(u) \leq \sup \gamma_e(u) \quad \forall\, u \in \mathbb{R}$, and define the maximal monotone multivalued function φ as in Fig. 2.

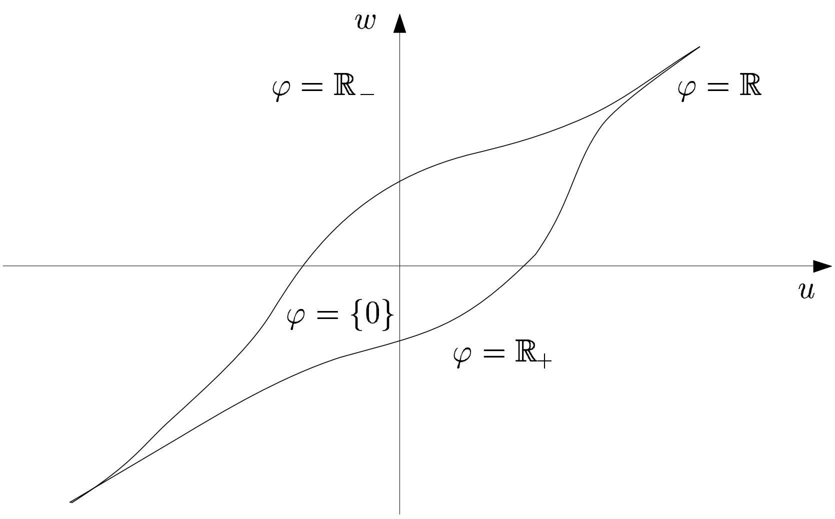

Figure 2: Generalized play operator

The functions γ_e and γ_r define a hysteresis which is called *generalized play operator.* This operator can be set in the form of a differential inclusion as in system (2.2) by $\dot{w} \in \varphi(u, w)$ in $[0, T]$, which is equivalent to the variational inequality ([35])

$$\begin{cases} u \in J(w) & := & [\inf \gamma_e^{-1}(w), \sup \gamma_r^{-1}(w)], \\ \dot{w}(u - v) & \geq & 0 \quad \forall v \in Jw. \end{cases}$$

By coupling this operator with a PDE, we get the problem

$$\begin{cases} \frac{\partial u}{\partial t} + \xi - \Delta u & = f & \text{in} & \Omega \times (0, T), \\ \frac{\partial w}{\partial t} - \xi & = 0 & \text{in} & \Omega \times (0, T), \\ \xi \in \varphi(u, w) & & \text{in} & \Omega \times (0, T), \\ u(\cdot, 0) = u_0, \quad w(\cdot, 0) & = w_0 & \text{in} & \Omega. \end{cases} \tag{2.4}$$

Here it is assumed that $\Omega \subset \mathbb{R}^n$ is smooth, $T > 0$,

$$f \in L^2(0, T; H^{-1}(\Omega)) \text{ and } u_0, w_0 \in L^2(\Omega).$$

By applying standard results of the theory of nonlinear semigroups it is shown in [35] that under certain additionally conditions the Cauchy-problem (2.4) has one and only one solution in the sense of Benilan. This solution depends continuously and monotonically on the data u_0, w_0 and f. It is also demonstrated in [35] that the shortly characterized approach can be extended to generalized *Prandtl-Ishlinskij operators of play type* which are used for the description of elasto-plastic material laws. □

Our aim is to investigate the properties of the inequality (2.1), (2.2) using only informations (measurements) from the observation operator (2.3). The concrete nonlinearities (contact laws and material laws) are assumed to be unknown. We consider them as *uncertainties* in the system. In order to describe the principal parts of such an uncertain dynamic system we use methods from absolute stability theory. The main idea is to characterize *a priori* information about the considered class of nonlinearities by means of suitably chosen quadratic forms. These quadratic forms, together with the description of the linear parts of system (2.1) – (2.3) in form of transfer functions "from an input to an output", give the necessary information for the construction of observers which are stable with respect to initial conditions in the considered class (2.1) – (2.3). The formal definition of the uncertainty parts of (2.1), (2.2) is the following one (see also [6, 18, 20, 21, 22]).

Definition 2.2 *a) Suppose F and G are quadratic forms on $Y_1 \times \Xi$. The* ***class of nonlinearities*** *$\mathcal{N}(F, G)$ defined by F and G consists of all maps $\varphi : \mathbb{R}_+ \times W \to 2^\Xi$ such that for any $y(\cdot) \in L^2_{\text{loc}}(0, \infty; Y_1)$ with $\dot{y}(\cdot) \in L^2_{\text{loc}}(0, \infty; Y_{-1})$ and any $\xi(\cdot) \in L^2_{\text{loc}}(0, \infty; \Xi)$ with $\xi(t) \in \varphi(t, Cy(t))$ for a.e. $t \geq 0$, it follows that $F(y(t), \xi(t)) \geq 0$ for a.e. $t \geq 0$ and (for any such pair $\{y, \xi\}$) there exists a continuous functional $\Phi : W \to \mathbb{R}$ such that for any times $0 \leq s < t$ we have* $\int_s^t G(y(\tau), \xi(\tau)) d\tau \geq \Phi(Cy(t)) - \Phi(Cy(s))$.

*b) The **class of functionals** $\mathcal{M}(d)$ defined by a constant $d > 0$ consists of all maps $\psi : Y_1 \to \mathbb{R}_+$ such that for any $y \in L^2_{\rm loc}(0,\infty;Y_0)$ with $\dot y \in L^2_{\rm loc}(0,\infty;Y_1)$ the function $t \mapsto \psi(y(t))$ belongs to $L^1(0,\infty;\mathbb{R})$ satisfying $\int\limits_0^\infty \psi(y(t))dt \le d$ and for any $\varphi \in \mathcal{N}(F,G)$ and any $\psi \in \mathcal{M}(d)$ the Cauchy-problem (2.1) $-$ (2.3) has a solution $\{y(\cdot),\xi(\cdot)\}$ on any time interval $[0,T]$.*

Remark 2.2 The functional Φ used in the description of the class $\mathcal{N}(F,G)$ can be considered as generalized potential of φ ([22]). Let us assume for a moment that $W = \Xi$ and $\varphi : W \to W$ is a continuous nonlinear mapping of gradient or potential type, i.e., there exists a continuous Fréchet-differentiable nonlinear functional $\Phi : W \to \mathbb{R}$, whose Fréchet derivative $\Phi'(w) \in \mathcal{L}(W,\mathbb{R})$ at any $w \in W$ can be represented in the form $\Phi'(w)\eta = (\varphi(w),\eta)$ for any $\eta \in W$. Then we can write for any function $w \in W^{1,2}_{\rm loc}(0,+\infty;W)$ and any times $0 \le s < t$ the path integral formula

$$\int_s^t (\dot w(\tau),\varphi(w(\tau))_W\, d\tau = \Phi(w(t)) - \Phi(w(s))\,.$$

It follows in this case that the properties described in Definition 2.2 a) are satisfied with respect to the quadratic form G if we take (with $W = \Xi$) the form

$$G(y,\xi) := \;(CAy + CB\xi,\xi)_\Xi\,. \tag{2.5}$$

□

3 Dissipativity with respect to observations

A basic tool for the derivation of sufficient conditions for absolute stability and instability of observations to variational inequalities is the following version of the *Frequency Domain Theorem* or *Kalman-Yakubovich-Popov lemma (KYP lemma)* ([6, 8, 18, 20, 25, 36]). In the infinite dimensional setting certain regularity assumptions are necessary which we formulate at the beginning of this section. In the next part of the paper up to theorem $3.1'$ it is assumed that all spaces and operators are complex.

(F1) The operator $A \in \mathcal{L}(Y_1, Y_{-1})$ is *regular* ([11, 20, 24]), i.e., for any $T > 0$, $y_0 \in Y_1, \psi_T \in Y_1$ and $f \in L^2(0,T;Y_0)$ the solutions of the direct problem

$$\dot y = Ay + f(t)\,,\; y(0) = y_0\,, \quad \text{a.a. } t \in [0,T]$$

and of the dual problem

$$\dot\psi = -A^*\psi + f(t)\,,\; \psi(T) = \psi_T\,, \quad \text{a.a. } t \in [0,T]$$

are strongly continuous in t in the norm of Y_1. Here (and in the following) $A^* \in \mathcal{L}(Y_{-1}, Y_0)$ denotes the adjoint to A, i.e., $(Ay,\eta)_{-1,1} = (y, A^*\eta)_{-1,1}\,,\ \forall y,\eta \in Y_1\,.$

Remark 3.1 The assumption **(F1)** is satisfied ([20]) if the embedding $Y_1 \subset Y_0$ is completely continuous, i.e., transforms bounded sets from Y_1 into compact sets in Y_0. □

(F2) The pair (A,B) is L^2*-controllable*, ([6, 18, 20]) i.e., for arbitrary $y_0 \in Y_0$ there exists a control $\xi(\cdot) \in L^2(0,\infty;\Xi)$ such that the problem

$$\dot{y} = Ay + B\xi\,, \quad y(0) = y_0$$

is well-posed on the semiaxis $[0,+\infty)$, i.e., there exists a solution $y(\cdot) \in \mathcal{L}_\infty$ with $y\,(0) = y_0$.

Remark 3.2 It is easy to see that a pair (A,B) is L^2-controllable if this pair is *exponentially stabilizable*, i.e., if an operator $K \in \mathcal{L}(Y_0,\Xi)$ exists such that the solution $y(\cdot)$ of the Cauchy-problem $\dot{y} = (A+BK)\,y,\ y(0) = y_0$, decreases exponentially as $t \to \infty$, i.e.,

$$\exists\, c > 0 \quad \exists\, \varepsilon > 0 : \ \|y(t)\|_0 \le c\, e^{-\varepsilon t}\, \|y_0\|_0\,, \quad \forall t \ge 0\,.$$

Note that the usual condition of *exact controllability* (on any finite time interval) is in general not satisfied for pairs of operators (A,B) arising from PDE problems ([34]). □

(F3) Let $F(y,\xi)$ be a Hermitian form on $Y_1 \times \Xi$, i.e.,

$$F(y,\xi) = (F_1 y, y)_{-1,1} + 2\,\mathrm{Re}\,(F_2 y,\xi)_\Xi + (F_3\,\xi,\xi)_\Xi,$$

where

$$F_1 = F_1^* \in \mathcal{L}(Y_1,Y_{-1})\,,\ F_2 \in \mathcal{L}(Y_0,\Xi)\,,\ F_3 = F_3^* \in \mathcal{L}(\Xi,\Xi)\,.$$

Define the *frequency-domain condition*

$$\alpha := \sup_{\omega,y,\xi} (\|y\|_1^2 + \|\xi\|_\Xi^2)^{-1} F(y,\xi)\,,$$

where the supremum is taken over all triples $(\omega,y,\xi) \in \mathbb{R}_+ \times Y_1 \times \Xi$ such that $i\omega y = Ay + B\xi$.

Theorem 3.1 *a) (Frequency Theorem for the Nonsingular Case, ([20])*
Assume for the linear operators $A \in \mathcal{L}(Y_1,Y_{-1}), B \in \mathcal{L}\,(\Xi,Y_{-1})$ *and the Hermitian form* F *on* $Y_1 \times \Xi$ *that the assumptions* **(F1)**, **(F2)** *are satisfied. Then there exist an operator* $P = P^* \in \mathcal{L}(Y_{-1},Y_0) \cap \mathcal{L}(Y_0,Y_1)$ *and a number* $\delta > 0$ *such that*

$$2\,\mathrm{Re}\,(Ay + B\xi, Py)_{-1,1} + F\,(y,\xi) \le -\delta(\|y\|_1^2 + \|\xi\|_\Xi^2)\,, \quad \forall (y,\xi) \in Y_1 \times \Xi\,, \qquad (3.1)$$

if and only if the frequency-domain condition from **(F3)** *with* $\alpha < 0$ *is satisfied.*

b) (Frequency Theorem for the Singular Case, [20])
Let the assumptions in Theorem 3.1a) be satisfied, and, in addition, let $B \in \mathcal{L}(\Xi,Y_0)$. *Then, for the existence of an operator* $P = P^* \in \mathcal{L}(Y_{-1},Y_0) \cap \mathcal{L}(Y_0,Y_1)$ *such that*

$$\mathrm{Re}\,(Ay + B\xi, Py)_{-1,1} + F\,(y,\xi) \le 0\,, \quad \forall\,(y,\xi) \in Y_1 \times \Xi\,,$$

it is necessary and sufficient that the following two conditions are fulfilled:

1) $\alpha \le 0$, *where* α *is from* **(F3)** ;

2) *The functional* $J(y(\cdot),\xi(\cdot)) := \int_0^\infty F(y(\tau),\xi(\tau))\,d\tau$ *is bounded from above on any set*

$$\mathfrak{M}_{y_0} := \{y(\cdot),\xi(\cdot) : \dot{y} = Ay + B\xi \text{ on } \mathbb{R}_+, y(0) = y_0\,,\ y(\cdot) \in \mathcal{L}_\infty\,,\ \xi(\cdot) \in L^2(0,\infty;\Xi)\}\,.$$

Remark 3.3 a) Let, in addition to the assumptions of Theorem 3.1b), A be the generator of a C_0-*group* on Y_0 and the pair $(A, -B)$ be L^2-controllable. Then the condition $\alpha \leq 0$, where α is from **(F3)**, is sufficient for the assertion of Theorem 3.1b) ([18, 20, 22]). Note that the existence of C_0-groups is given for conservative wave equations, plate problems, and other important PDE classes ([12]).

b) Under the assumptions of Theorem 3.1 a) there exist operators $P = P^* \in \mathcal{L}(Y_0, Y_0), L \in \mathcal{L}(\Xi, Y_0)$ and $K = K^* \in \mathcal{L}(\Xi, \Xi)$ such that

$$2\,\mathrm{Re}\,(Ay + B\xi, Py)_{-1,1} + F\,(y, \xi) = -\|L^* y - K\xi\|_{\Xi}^2, \quad \forall (y, \xi) \in Y_1 \times \Xi. \tag{3.2}$$

This equality is equivalent to the system of *Lur'e* or *algebraic Riccati equations*

$$A^* P + PA + F_1 = -LL^*\,,\ PB + F_2 = LK\,,\ F_3 = -K^* K\,. \tag{3.3}$$

Let us formulate Theorem 3.1a) in terms of operator symbols. For this we construct (see also [2]) formally for the given pseudodifferential operators $A(\mathbb{D}), B(\mathbb{D}), F_1(\mathbb{D}), F_2(\mathbb{D})$ and $F_3(\mathbb{D})$ with $\mathbb{D} = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_m}\right)$ their full symbols. Thus we get for λ from the domain $\mathcal{G} \subset \mathbb{R}^m$ the families of analytic *matrix functions* $\hat{A}(\lambda), \hat{B}(\lambda), \hat{F}_1(\lambda) = \hat{F}_1^*(\lambda)$ and $\hat{F}_3(\lambda) = \hat{F}_3^*(\lambda)$ of dimensions $n \times n$, $n \times k$, $n \times n$, $n \times k$ and $k \times k$, respectively, and the Hermitian form

$$\hat{F}\,(\lambda, y, \xi) = y^* \hat{F}_1(\lambda) y + 2\,\mathrm{Re}\, y^* \hat{F}_2(\lambda)\xi + \xi^* \hat{F}_3(\lambda)\xi$$

on $\mathbb{C}^n \times \mathbb{C}^k$. Now the exponential stabilizibility of (A, B) can be characterized by means of the pair of matrix functions $\left(\hat{A}(\lambda), \hat{B}(\lambda)\right)$: This pair is assumed to be *stabilizable* in $\mathcal{G}$, i.e., there exists a matrix-valued function $\hat{S}(\lambda)$ such that for all $\lambda \in \mathcal{G}$ the inclusion

$$\sigma\left(\hat{A}(\lambda) + \hat{B}(\lambda)\hat{S}(\lambda)\right) \subset \{s \in \mathbb{C} : \mathrm{Re}\, s < -\varepsilon\}$$

with some $\varepsilon > 0$ is satisfied.
We introduce now the matrix transfer function $\chi(\omega, \lambda) = (i\omega I_n - \hat{A}(\lambda))^{-1} B(\lambda)$. Suppose that $\Omega \subset \mathbb{R}^{m+1}$ is the analyticity domain of $\chi(\omega, \cdot)$ and let $\Pi(\omega, \lambda)$ be the $k \times k$-Hermitian matrix of the form $\hat{F}(\lambda, \chi(\omega, \lambda)\xi, \xi)$. Now, the Frequency Theorem 3.1a) can be formulated in the following version which is important for practical applications since all conditions for the existence of certain operators for the infinite-dimensional case are stated for matrix functions.

Theorem 3.1' ([19]) *Let the pair* $(\hat{A}(\lambda), \hat{B}(\lambda))$ *be stabilizable in* $\mathcal{G} = \mathrm{Pr}_n \Omega$ *and suppose that there exists a* $\delta > 0$ *such that*

$$\Pi(\omega, \lambda) \leq -\delta I \quad \textit{for all} \quad (\omega, \lambda) \in \Omega.$$

Then there exist symbols $\hat{P}(\lambda), \hat{L}(\lambda)$ *and* $\hat{K}(\lambda)$ *and associated pseudodifferential operators* $P(\mathbb{D}), L(\mathbb{D})$ *and* $K(\mathbb{D})$ *satisfying the Lur'e equations*

$$\begin{cases} A^*(\mathbb{D})P(\mathbb{D}) + P(\mathbb{D})A(\mathbb{D}) + F_1(\mathbb{D}) = -L(\mathbb{D})L^*(\mathbb{D})\,, \\ P(\mathbb{D})B(\mathbb{D}) + F_2(\mathbb{D}) = L(\mathbb{D})K(\mathbb{D}),\ F_3(\mathbb{D}) = -K(\mathbb{D})K^*(\mathbb{D})\,. \end{cases} \tag{3.4}$$

□

Let us state the following theorem which is based on the frequency-domain approach and which can be considered as generalization of energy-type equalities for PDE's. With the superscript ^{c} we denote the complexification of spaces and operators and the extension of quadratic forms to Hermitian forms.

Theorem 3.2 *Consider the evolution problem* (2.1) − (2.3) *with* $\varphi \in \mathcal{N}(F,G)$ *and* $\psi \in \mathcal{M}(d)$. *Suppose that for the operators* A^c, B^c *the assumptions* **(F1)** *and* **(F2)** *are satisfied. Suppose also that there exist an* $\alpha > 0$ *such that with the transfer operator*

$$\chi^{(z)}(s) = D^c(sI^c - A^c)^{-1}B^c + E^c \qquad (s \notin \sigma(A^c)) \tag{3.5}$$

the frequency-domain condition

$$F^c\left((i\omega I^c - A^c)^{-1}B^c\xi, \xi\right) + G^c\left((i\omega I^c - A^c)^{-1}B^c\xi, \xi\right) \le -\alpha\|\chi^{(z)}(i\omega)\xi\|^2_{Z^c} \qquad \forall \omega \in \mathbb{R} : i\omega \notin \sigma(A^c)\,, \quad \forall \xi \in \Xi^c \tag{3.6}$$

is satisfied and the functional

$$J(y(\cdot),\xi(\cdot)) := \int_0^\infty \left[F^c(y(\tau),\xi(\tau)) + G^c(y(\tau),\xi(\tau)) + \alpha\|D^c y(\tau) + E^c\xi(\tau)\|^2_{Z^c}\right] d\tau$$

is bounded from above on any set $\mathfrak{M}_{y_0}$ *defined in Theorem 3.1b).*
Then there exist an (real) operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ *and a number* $\delta > 0$ *such that with the Lyapunov-functional* $V(y) := (y, Py)_0 \quad (y \in Y_0)$ *for an arbitrary solution* $\{y(\cdot), \xi(\cdot)\}$ *of (2.1), (2.2) with* $\varphi \in \mathcal{N}(F,G)$ *and* Φ *as generalized potential of* φ, *the inequality*

$$V(y(t)) - V(y(s)) + \Phi\left(Cy(t)\right) - \Phi\left(Cy(s)\right) + \int_s^t F(y(\tau),\xi(\tau))d\tau + \int_s^t \left[\psi(y(\tau)) - \psi(-Py(\tau) + y(\tau))\right] d\tau + \delta\int_s^t \|Dy(\tau) + E\xi(\tau)\|^2_Z\, d\tau \le \int_s^t (f(\tau), Py(\tau))_{-1,1} d\tau \tag{3.7}$$

is satisfied on any time interval $0 \le s < t$.

Proof For the Hermitian form $F^c + G^c$ all assumptions of the Frequency Theorem 3.1b) are satisfied. It follows ([20]) that there exist a (real) operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and a number $\delta > 0$ such that

$$(-Ay - B\xi, Py)_{-1,1} \ge F(y,\xi) + G(y,\xi) + \delta\|Dy + E\xi\|^2\,, \qquad \forall (y,\xi) \in Y_1 \times \Xi\,. \tag{3.8}$$

Consider the inequality (2.1) with the given solution $\{y(\cdot), \xi(\cdot)\}$. Take the special test function $P\eta$ to see that on an arbitrary time interval $[0,T]$

$$(\dot y(t) - Ay(t) - B\xi(t) - f(t), P\eta - y(t))_{-1,1} + \psi(P\eta) - \psi(y) \ge 0\,, \qquad \forall \eta \in Y_1\,, \text{a.a.}\, t \in [0,T]\,. \tag{3.9}$$

Substitute now $P\eta(t) = -Py(t) + y(t)$. Then (3.9) can be written as

$$(\dot{y}(t) - Ay(t) - B\xi(t) - f(t), -Py(t))_{-1,1} + \psi(-Py(t) + y(t)) - \psi(y(t)) \geq 0\,, \quad \text{a.a. } t \in [0,T] \tag{3.10}$$

or

$$\begin{aligned} &(\dot{y}(t)\,, Py(t))_{-1,1} - (Ay(t) + B\xi(t)\,, Py(t))_{-1,1} \\ &-\psi\,(-Py(t) + y(t)) + \psi(y(t)) \leq (f(t), Py(t))_{-1,1}\,, \qquad \text{a.a. } t \in [0,T]\,. \end{aligned} \tag{3.11}$$

Now we use the inequality (3.8) in order to estimate

$$\begin{aligned} &-(Ay(t) + B\xi(t), P\xi(t))_{-1,1} \\ &\geq F(y(t), \xi(t)) + G(y(t), \xi(t)) + \delta \|Dy(t) + E\xi(t)\|_Z^2\,, \qquad \text{a.a. } t \in [0,T]\,. \end{aligned} \tag{3.12}$$

We get from (3.11) and (3.12)

$$\begin{aligned} &(\dot{y}(t), Py(t))_{-1,1} + F(y(t), \xi(t)) + G(y(t), \xi(t)) \\ &+ \psi(y(t)) - \psi(-Py(t) + y(t)) + \delta \|Dy(t) + E\xi(t)\|_Z^2 \leq (f(t), Py(t))_{-1,1}\,, \qquad \text{a.a. } t \in [0,T]\,. \end{aligned} \tag{3.13}$$

If we take in the inequality (3.13) the integral on an arbitrary time interval $0 \leq s < t$ we receive

$$\begin{aligned} &V(y(t)) - V(y(s)) + \int_s^t F(y(\tau), \xi(\tau))d\tau + \int_s^t G(y(\tau)\,, \xi(\tau))d\tau \\ &+ \int_s^t [\psi(y(\tau)) - \psi(-Py(\tau) + y(\tau))]\, d\tau \;+\; \delta \int_s^t \|Dy(\tau) + E\,\xi(\tau)\|_Z^2\, d\tau \\ &\leq \int_s^t (f(\tau), Py(\tau))_{-1,1} d\tau\,. \end{aligned} \tag{3.14}$$

Now it remains to use the properties of G with respect to the solution $\{y(\cdot), \xi(\cdot)\}$, i.e.,

$$\int_s^t G(y(\tau), \xi(\tau))d\tau \geq \Phi\,(Cy(t)) - \Phi(Cy(s))\,. \tag{3.15}$$

From (3.14) and (3.15) the assertion of Theorem 3.2) follows immediately. ■

Remark 3.4 Inequality (3.7) can be considered ([7, 11, 20, 22, 31, 32, 37]) as generalized *energy balance inequality* or *dissipation inequality.* For a nonnegative operator P the term $V(y(t))$ in (3.7) is the *energy stored in the state* $y(t)$. The form $-F$ which reflects the influence of the constitutive law can be considered as *energy supply rate* and the integral $-\int_s^t F(y(\tau), \xi(\tau))d\tau$ as *energy* absorbed by the system in the time period $[s,t]$. A *contact energy* term ([11, 14, 15, 26, 29]) in (3.7) is characterized by ψ. □

The physical interpretation of inequality (3.7) leads to the following definition.

Definition 3.1 *The Cauchy problem (2.1), (2.2) is called* ***dissipative in the sense of Willems*** *([37]) with respect to the observation* z *from (2.3) and the classes* $\mathcal{N}(F,G)$ *and* $\mathcal{M}(d)$ *if there exist a* $\delta > 0$ *and a bounded self-adjoint operator* $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ *such that for any solution* $\{y(\cdot), \xi(\cdot)\}$ *of (2.1), (2.2) and any time interval* $0 \le s < t$ *the inequality (3.7) holds.*

In other words we get from Theorem 3.2 the following

Corollary 3.1 *Under the assumptions of Theorem 3.2 the Cauchy problem (2.1), (2.2) is dissipative in the sense of Willems with respect to the observation* z *from (2.3) and the classes* $\mathcal{N}(F,G)$ *and* $\mathcal{M}(d)$.

4 Absolute observation - stability and instability of evolutionary inequalities

We continue the investigation of energy like properties for the observation operators from the inequality problem (2.1), (2.2) with $f \equiv 0$.

The next definition generalizes the concepts which are introduced in [20, 21, 22, 38] for output operators of evolution equations, namely in extending them to the observation operators of a class of evolutionary variational inequalities. In the following we denote for a function $z(\cdot) \in L^2(\mathbb{R}_+; Z)$ their norm by

$$\|z(\cdot)\|_{2,Z}^2 := \int_0^\infty \|z(t)\|_Z^2 \, dt \, .$$

Definition 4.1 *a) The inequality (2.1), (2.2) is said to be* ***absolutely dichotomic*** *(i.e., in the classes* $\mathcal{N}(F,G), \mathcal{M}(d)$*)* ***with respect to the observation*** z *from (2.3) if for any solution* $\{y(\cdot), \xi(\cdot)\}$ *of (2.1), (2.2) with* $y(0) = y_0$, $\xi(0) = \xi_0 \in \mathcal{E}(y_0)$ *the following is true: Either* $y(\cdot)$ *is unbounded on* $[0, \infty)$ *in the* Y_0*-norm or* $y(\cdot)$ *is bounded in* Y_0 *in this norm and there exist constants* c_1 *and* c_2 *(which depend only on* $A, B, \mathcal{N}(F,G)$ *and* $\mathcal{M}(d)$*) such that*

$$\|Dy(\cdot) + E\xi(\cdot)\|_{2,Z}^2 \le c_1(\|y_0\|_0^2 + c_2) \, . \tag{4.1}$$

b) The inequality (2.1), (2.2) is said to be ***absolutely stable with respect to the observation*** z *from (2.3) if (4.1) holds for any solution* $\{y(\cdot), \xi(\cdot)\}$ *of (2.1), (2.2).*

c) The inequality (2.1), (2.2) is said to be ***absolutely unstable with respect to the observation*** z *from (2.3) if for any constants* c_1 *and* c_2 *in (4.1) there exist initial states* $y_0 \in Y_0$ *and* $\xi_0 \in \mathcal{E}(y_0)$ *such that solutions* $\{y(\cdot), \xi(\cdot)\}$ *of (2.1), (2.2) starting in these points do not satisfy (4.1) .*

Our first result concerns frequency-domain conditions for absolute observation dichotomy of controlled evolutionary variational inequalities.

Theorem 4.1 *Suppose that the assumptions of Theorem 3.2 are satisfied with* $f \equiv 0$. *Assume additionally that any potential* Φ *from the class* $\mathcal{N}(F,G)$ *is nonnegative. Suppose also that for any potential* Φ *there exists a constant* $c > 0$ *such that* $\Phi(Cy) \le c\,\|y\|_0^2\,, \quad \forall y \in Y_0$. *Then the inequality (2.1), (2.2) is absolutely dichotomic with respect to the observation* z *from (2.3).*

Proof As was shown in the proof of Theorem 3.2 under the assumptions of the present theorem there exist an operator $P = P^* \in \mathcal{L}(Y_0, Y_0)$ and a number $\delta > 0$ such that for an arbitrary solution $\{y(\cdot), \xi(\cdot)\}$ of (2.1), (2.2) the inequality (3.7) holds. Suppose that $y(\cdot)$ is bounded in Y_0 on $[0,\infty)$. Define the function

$$
\begin{aligned}
W(t) : \; &= \; V(y(t)) + \Phi(Cy(t)) + \int_0^t F(y(\tau), \xi(\tau))d\tau \\
&+ \int_0^t \left[\psi(y(\tau)) - \psi(-Py(\tau) + y(\tau))\right] d\tau .
\end{aligned}
$$

From (3.7) it follows that for arbitrary s, t such that $0 \le s < t$

$$
W(t) - W(s) \le -\delta \int_s^t \|Dy(\tau) + E\xi(\tau)\|_Z^2 \, d\tau \le 0\,, \tag{4.2}
$$

i.e., W is monotonically decreasing. Since $y(\cdot)$ is bounded in Y_1 on $[0,\infty)$ the function $W(\cdot)$ is bounded from below. It follows that there exists the limit $\lim\limits_{t\to+\infty} W(t)$ and that for any $t > 0$ by (4.2) the inequality

$$
\delta \int_0^t \|Dy(\tau) + E\,\xi(\tau)\|_Z^2 \, d\tau \le W(0) - \lim_{t\to\infty} W(t) \le c\,\|y_0\|_0^2
$$

is true. From this we get immediately that $z = Dy + E\,\xi \in L^2(0,\infty; Z)$. Suppose now that $W(t) \to -\infty$ for $t \to \infty$. Then from $\Phi(w) \ge 0$, $\int_0^t F(y(\tau),\, \xi(\tau))\, d\tau \ge 0$ and the boundedness of $\int_0^t [\psi(y(\tau)) - \psi(-Py(\tau) + y(\tau))]d\tau$ it follows that $V(y(t)) \to -\infty$ for $t \to +\infty$. From this we conclude that $\|y(t)\|_0 \to \infty$. ∎

In order to get absolute stability properties of (2.1), (2.2) with respect to the observation z of (2.3) we need an assumption for (2.1) – (2.3) which is called "minimal stability" for the class of evolution equations in [21, 22].

Definition 4.2 *The inequality (2.1)–(2.3) with* $f \equiv 0$ *is said to be* ***minimally stable*** *if the resulting equation for* $\psi \equiv 0$ *is minimally stable, i.e., there exists a bounded linear operator* $K : Y_1 \to \Xi$ *such that the operator* $A + BK$ *is stable, i.e.,*

$$
\sigma(A+BK) \subset \{s \in \mathbb{C} : \operatorname{Re} s \le -\varepsilon < 0\} \quad \textit{with} \quad F(y,Ky) \ge 0\,, \qquad \forall y \in Y_1\,, \tag{4.3}
$$

and

$$\int_s^t G(y(\tau)\,,\ Ky(\tau))d\tau \geq 0\,, \quad \forall\, s,t : 0 \leq s < t\,, \quad \forall y \in L^2_{\mathrm{loc}}(\mathbb{R}_+; Y_1)\,. \tag{4.4}$$

Theorem 4.2 *Suppose that the assumptions of Theorem 4.1 are satisfied and the inequality (2.1)–(2.3) with* $f \equiv 0$ *is minimally stable, i.e., (4.3) and (4.4) are satisfied with some operator* $K \in \mathcal{L}(Y_1, \Xi)$. *Suppose also that the pair* $(A+BK, D+EK)$ *is observable in the sense of Kalman ([6]), i.e., for any solution* $y(\cdot)$ *of*

$$\dot{y} = (A + BK)y\,, \quad y(0) = y_0\,,$$

with $z(t) = (D + EK)y\,(t) = 0$ *for a.a.* $t \geq 0$ *it follows that* $y(0) = y_0 = 0$.
Then inequality (2.1), (2.2) is absolutely stable with respect to the observation z *from (2.3).*

Proof Under the assumptions of the given theorem there exist by Theorem 3.1b) a (real) operator $P = P^* \in \mathcal{L}\,(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and a number $\delta > 0$ such that the inequality (3.8) is satisfied. Setting in (3.8) $\xi = Ky$ from (4.3) with arbitrary $y \in Y_1$ we get with (4.4) the inequality

$$((A + BK)y\,,\, Py)_{-1,1} \leq -\delta\,\|Dy + EKy\,\|^2_Z\,, \quad \forall\, y \in Y_1\,. \tag{4.5}$$

Using the fact that $A + BK$ is a stable operator and the pair $(A + BK, D + EK)$ is observable, it follows ([6, 9, 36]) from (4.5) that $P = P^* \geq 0$. Suppose now that $\{y(\cdot), \xi(\cdot)\}$ is an arbitrary solution of (2.1), (2.2) with $f \equiv 0$. With the Lyapunov-functional $V(y) = (y, Py)_0 \geq 0$ it follows from (3.7) that for arbitrary $t \geq 0$

$$-V(y_0) - \Phi\,(Cy_0) + \int_0^t [\psi(y(\tau)) - \psi(-Py(\tau) + y(\tau))]d\tau + \delta \int_0^t \|Dy(\tau) + E\,\xi(\tau)\|^2_Z\, d\tau \leq 0\,. \tag{4.6}$$

Since by assumption $\int_0^t [\psi(y(\tau) - \psi(-Py(\tau) + y(\tau))]d\tau \geq -c_2 > -\infty$ we get from (4.6) for arbitrary $t \geq 0$ the inequality

$$\delta \int_0^t \|Dy(\tau) + E\,\xi(\tau)\|^2_Z\, d\tau \leq V(y_0) + \Phi(Cy_0) + c_2 \leq V(y_0) + c\,\|y_0\|^2_0 + c_2\,. \tag{4.7}$$

The property (4.7) implies now the estimate (4.1) . ■

In the next theorem we show that under certain assumptions the inequality (2.1) –(2.3) is absolutely unstable with respect to the observation.

Definition 4.3 *The inequality (2.1)–(2.3) with $f \equiv 0$ is said to be* ***minimally unstable*** *if the resulting equation for $\psi \equiv 0$ is minimally unstable, i.e., there exists a bounded linear operator $K : Y_1 \to \Xi$ such that the operator $A + BK$ is unstable, i.e.,*

$$\sigma(A + BK) \cap \{s \in \mathbb{C} : \operatorname{Re} s \geq \varepsilon > 0\} \neq \varnothing \quad \textit{with} \quad F(y, Ky) \geq 0\,, \quad \forall y \in Y_1\,, \tag{4.8}$$

and

$$\int_s^t G(y(\tau)\,,\, Ky(\tau))d\tau \geq 0\,, \quad \forall\, s, t : 0 \leq s < t\,, \forall y \in L^2_{\mathrm{loc}}(\mathbb{R}_+; Y_1)\,. \tag{4.9}$$

Theorem 4.3 *Suppose that the assumptions of Theorem 4.1 are satisfied and the inequality (2.1)–(2.3) with $f \equiv 0$ is minimally unstable, i.e., (4.8) and (4.9) are satisfied with some operator $K \in \mathcal{L}(Y_1, \Xi)$. Suppose also that for the equation*

$$\dot{y} = (A + BK)y\,,\ y(0) = y_0\,,$$

in Y_0 there exists a splitting $Y_0 = Y_0^+ \oplus Y_0^-$ such that for any $y_0 \in Y_0^+$ the solution $y(\cdot)$ of this equation satisfies $\lim_{t\to\infty} y(t) = 0$ and for any $y_0 \in Y_0^-$ a unique solution $y(\cdot)$ of the last equation exists on $(-\infty, 0)$ satisfying $\lim_{t\to-\infty} y(t) = 0$. Assume that the pair $(A+BK, D)$ is observable on Y_0^+, i.e., for any solution $y(\cdot)$ with $z(t) = Dy(t) = 0$ for a.a. $t \geq 0$ it follows that $y(0) = y_0 = 0$. Then there exists an operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$, which is non-negative on Y_0^+ and negative on Y_0^-. If $y_0 \in Y_0^-$ is an arbitrary point satisfying

$$\frac{1}{2}(y_0, Py_0)_0 + \Phi(Cy_0) < -d\,, \tag{4.10}$$

where d is the parameter from the class $\mathcal{N}(d)$, then for any solution $\{y(\cdot), \xi(\cdot)\}$ of (2.1)– (2.3) with $y(0) = y_0$ the term $\|Dy(\cdot)\|^2_{Z,2}$ is unbounded on $[0, \infty)$ proposed that $D : Y_1 \to Z$ is invertible. It follows that the inequality (2.1)– (2.3) is absolutely unstable with respect to the observation $z(\cdot) = Dy(\cdot)$.

Proof Under the assumptions of the present theorem we get on the basis of Theorem 3.1b) the existence of an operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and a number $\delta > 0$ such that inequality (3.8) with $E = 0$ is satisfied. Setting again in (3.8) $\xi = Ky$ from (4.8) with arbitrary $y \in Y_1$ we see that inequality (4.5) with $E = 0$ is true. Using now the fact that $A + BK$ is an unstable operator, the pair $(A + BK, D)$ is observable and that there exists a splitting $Y_0 = Y_0^- \bigoplus Y_0^+$ with the above properties, we see ([6, 9, 20]) that the Lyapunov-functional $V(y) := (y, Py)_0$ is non-negative on Y_0^+ and negative on Y_0^-. Consider now an arbitrary solution $\{y(\cdot), \xi(\cdot)\}$ of (2.1), (2.2) with initial state $y_0 \in Y_0^-$. It follows that $V(y_0) < 0$ and, on the basis of (3.7),

$$V(y(t)) - \Phi\,(Cy_0) + \int_0^t [\psi(y(\tau)) - \psi(-Py(\tau) + y(\tau))]\,d\tau + \delta \int_0^t \|Dy(\tau)\|_Z^2\,d\tau \le V(y_0) < 0 \tag{4.11}$$

for all $t \ge 0$.
From (4.11) we conclude that for $t \to +\infty$

$$W(t) := V(y(t)) + \int_0^t [\psi(y(\tau)) - \psi(-Py(\tau) + y(\tau))]\,d\tau \not\to 0\,. \tag{4.12}$$

Indeed, if we suppose that (4.12) does not hold, we get from (4.11) for large t the estimate

$$\Phi(Cy_0) + \delta \int_0^t \|Dy(\tau)\|_Z^2\,d\tau \le \frac{1}{2} V\,(y_0) < 0\,,$$

which is impossible since $\Phi(Cy_0) \ge 0$ and $\delta > 0$.
Let us now show that

$$W(t) \to -\infty \quad \text{for} \quad t \to +\infty\,. \tag{4.13}$$

Suppose that (4.13) is not true. This means that there exists a constant $c > 0$ such that

$$W(t) \ge -c \quad , \quad \forall\, t \ge 0\,. \tag{4.14}$$

From (4.11) and (4.14) it follows now that for all $t \ge 0$

$$\delta \int_0^t \|\, Dy(\tau)\|_Z^2\,d\tau \le c + \Phi\,(Cy_0) + V\,(y_0)\,. \tag{4.15}$$

Since D is invertible, the inequality (4.14) shows that $y(\cdot) \in L^2(0,\infty;Y_1)$ and, consequently,

$$V\,(y(t)) \to 0 \quad \text{for} \quad t \to +\infty\,. \tag{4.16}$$

Recall now that for all $t \ge 0$

$$-\int_0^t [\psi(y(\tau)) - \psi(-Py(\tau) + y(\tau))]\,d\tau \le d\,. \tag{4.17}$$

It follows from (4.11), (4.16) and (4.17) that for sufficiently large t

$$0 \le \Phi\,(Cy_0) + d + \frac{1}{2} V\,(y_0)\,. \tag{4.18}$$

It is clear that (4.18) contradicts (4.10) . ■

5 Application of observation stability to the beam equation

Example 5.1 Consider the equation of a beam of length l, with damping and Hookean material, given as

$$\rho\, A\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\frac{EA}{3}\tilde{g}\left(\frac{\partial u}{\partial x}\right)\right) = 0\,, \tag{5.1}$$

$$u(0,t) = u(l,t) = 0 \quad \text{for} \quad t > 0\,, \tag{5.2}$$

$$u(x,0) = u_0(x)\,,\; u_t(x,0) = u_1(x) \quad \text{for} \quad x \in (0,l)\,. \tag{5.3}$$

Here u is the deformation in the x direction. Assume that the cross section area A, the viscose damping γ, the mass density ρ and the generalized modulus of elasticity E are constant. The nonlinear stress-strain law $\tilde{g}$, is given by

$$\tilde{g}(w) = 1 + w - (1+w)^{-2}\,, \quad w \in (-1,1)\,. \tag{5.4}$$

Let us break the stress-strain law into the sum of a linear term and a nonlinear term as $\tilde{g}(w) = g(w) + w$. Then the above model (5.1) can be rewritten as

$$\rho A\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{EA}{3}\frac{\partial u}{\partial x}\right) + \gamma\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\frac{EA}{3}g\left(\frac{\partial u}{\partial x}\right)\right) = 0\,. \tag{5.5}$$

Assume the Gelfand triple $\mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1}$ with

$$\mathcal{V}_0 := L^2(0,l)\,, \quad \mathcal{V}_1 := H_0^1(0,l) \quad \text{and} \quad \mathcal{V}_{-1} := H^{-1}(0,l)\,. \tag{5.6}$$

Then equation (5.1) $-$ (5.3) can be rewritten in $\mathcal{V}_{-1}$ as

$$\rho A u_{tt} + \mathcal{A}_1 u + \mathcal{A}_2 u_t + \mathcal{C}^* g(\mathcal{C}u) = 0\,, \tag{5.7}$$

$$u(0) = u_0\,, \quad u_t(0) = u_1\,, \tag{5.8}$$

with $\mathcal{A}_1 \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_{-1})$, $\mathcal{A}_2 \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_{-1})$ (strong damping), $\mathcal{C} \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)$ and $g : \mathcal{V}_0 \to \mathcal{V}_0$. The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are associated with their bilinear forms $a_i : \mathcal{V}_1 \times \mathcal{V}_1 \to \mathbb{R}$ $(i = 1,2)$ through $(\mathcal{A}_i v, w)_{\mathcal{V}_{-1},\mathcal{V}_1} = a_i(v,w)\,,\ \forall\, v, w \in \mathcal{V}_0\,.$
In order to get a variational interpretation of (5.7), (5.8) we make the following assumptions ([3, 4]) :

(A1)

a) The form a_1 is symmetric on $\mathcal{V}_0 \times \mathcal{V}_;$
b) a_1 is $\mathcal{V}_1$ continuous, i.e., for some $c_1 > 0$ holds $|a_1(v,w)| \le c_1\|v\|_{\mathcal{V}_1}\|w\|_{\mathcal{V}_1}$, $\forall\, v, w \in \mathcal{V}_1\,;$
c) a_1 is strictly $\mathcal{V}_1$-elliptic, i.e., for some $k_1 > 0$ holds $a_1(v,v) \ge k_1\|v\|^2_{\mathcal{V}_1}\,, \quad \forall\, v \in \mathcal{V}_1\,.$

(A2)

a) The form a_2 is $\mathcal{V}_1$ continuous, i.e., for some $c_2 > 0$ holds $|a_2(v,w)| \le c_2 \|v\|_{\mathcal{V}_1} \|w\|_{\mathcal{V}_1}$, $\forall\, v, w \in \mathcal{V}_1$.

b) The form a_2 is $\mathcal{V}_1$ coercive and symmetric, i.e., there are $k_2 > 0$ and $\lambda \ge 0$ s.t.

$$a_2(v,v) + \lambda_0 \|v\|^2_{\mathcal{V}_0} \ge k_2 \|v\|^2_{\mathcal{V}_1} \qquad \text{and}$$
$$a_2(v,w) = a_2(w,v)\,, \quad \forall v, w \in \mathcal{V}_1\,.$$

(A3)

a) The operator $\mathcal{C} \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)$ satisfies with some $k \ge 0$ the inequality $\|\mathcal{C}v\|_{\mathcal{V}_0} \le \sqrt{k}\|v\|_{\mathcal{V}_1}\,, \quad \forall\, v \in \mathcal{V}_1$.
$g : \mathcal{V}_0 \to \mathcal{V}_0$ is continuous and $\|g(v)\|_{\mathcal{V}_0} \le c_1\|v\|_{\mathcal{V}_0} + c_2$ for $v \in \mathcal{V}_0$, where c_1 and c_2 are nonnegative constants.

b) g is of gradient type, i.e., there exists a coninuous Frechét-differentiable functional $G : \mathcal{V}_0 \to \mathbb{R}$, whose Frechét derivative $G'(v) \in \mathcal{L}(\mathcal{V}_0, \mathbb{R})$ at any $v \in \mathcal{V}_0$ can be represented in the form

$$G'(v)w = (g(v), w)_{\mathcal{V}_0}\,, \quad \forall\, w \in \mathcal{V}_0\,.$$

c) $g(0) = 0$ and for some $\varepsilon < 1$ we have for all $v, w \in \mathcal{V}_0$

$$(g(v) - g(w), v - w)_{\mathcal{V}_0} \ge -\varepsilon k_1 k^{-1} \|v - w\|^2_{\mathcal{V}_0}\,. \tag{5.9}$$

We say that $u \in \mathcal{L}_T$ is a weak solution of (5.7), (5.8) if

$$(u_{tt}, \eta)_{\mathcal{V}_{-1},\mathcal{V}_1} + a_1(u,\eta) + a_2(u_t,\eta) + (g(\mathcal{C}u), \mathcal{C}u)_0 = 0 \qquad \forall \eta \in \mathcal{L}_T\,,\ \text{a.a. } t \in [0,T]\,. \tag{5.10}$$

Let us formulate our problem (5.10) in first order form on the energetic space $Y_0 := \mathcal{V}_1 \times \mathcal{V}_0$ in the coordinates $y = (y_1, y_2) = (u, u_t)$. Define for this $Y_1 := \mathcal{V}_1 \times \mathcal{V}_1$ and $a : Y_1 \times Y_1 \to \mathbb{R}$ by

$$a((v_1,v_2),(w_1,w_2)) = (v_2, w_1)_{\mathcal{V}_1} - a_1(v_1,w_2) - a_2(v_2,w_2)\,, \tag{5.11}$$
$$\forall\, (v_1,v_2),(w_1,w_2) \in Y_1 \times Y_1\,.$$

The norms in the product spaces Y_0 and Y_1 are given in the standard way by

$$\|(y_1,y_2)\|_0^2 := \|y_1\|^2_{\mathcal{V}_1} + \|y_2\|^2_{\mathcal{V}_0}\,, \quad (y_1,y_2) \in Y_0\,, \text{and}$$
$$\|(y_1,y_2)\|_1^2 := \|y_1\|^2_{\mathcal{V}_1} + \|y_2\|^2_{\mathcal{V}_1}\,, \quad (y_1,y_2) \in Y_1\,.$$

Then (5.10) can be rewritten as

$$(\dot{y}, \eta)_{-1,1} - a(y,\eta) = (B\varphi(Cy), \eta)_{-1,1}\,,\ y(0) = (u_0, u_1)\,, \quad \forall\, \eta \in Y_1\,, \tag{5.12}$$

where

$$B\varphi(Cy) := \begin{pmatrix} 0 \\ -\mathcal{C}^* g\,(\mathcal{C}y_1) \end{pmatrix}\,. \tag{5.13}$$

We can also write (5.12), (5.13) formally in the operator form

$$\dot{y} = Ay + B\varphi(Cy),\ y(0) = y_0\,, \tag{5.14}$$

where A is defined by

$$a(v,w) = (Av,w)_{-1,1}\,, \quad \forall\, v,w \in Y_1\,,$$

i.e.,

$$A = \begin{bmatrix} 0 & I \\ -\mathcal{A}_1 & -\mathcal{A}_2 \end{bmatrix}.$$

It is shown in [3, 4, 10, 30] that the embedding $Y_1 \subset Y_0$ is completely continuous and the operator A generates an analytic semigroup on Y_1, Y_0 and $Y_{-1} = \mathcal{V}_1 \times \mathcal{V}_{-1}$. Furthermore, its semigroup is exponentially stable on Y_1, Y_0 and Y_{-1}. From this it follows that the pair (A,B) is exponentially stabilizable. Let us consider with parameters $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ a more simplified form of (5.1) $-$ (5.3) written as

$$\frac{\partial^2 u}{\partial t^2} + 2\varepsilon\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = -\alpha\left(\frac{\partial}{\partial x}\left(-g\left(\frac{\partial u}{\partial x}\right)\right)\right) =: \alpha\frac{\partial}{\partial x}\,\xi \tag{5.15}$$

together with the boundary and initial conditions (5.2), (5.3), where we have $\xi = -g = \varphi$ introduced as new nonlinearity. According to (5.9) in **(A3)**a) we can assume that $\varphi \in \mathcal{N}(F)$ with the quadratic form $F(w,\xi) = \mu w^2 - \xi w$ on $\mathbb{R}\times\mathbb{R}$, where $\mu > 0$ is a certain parameter. Note that it is possible to include a second quadratic form G if we use the information from **(A3)**b).
Suppose that $\lambda_k > 0$ and e_k $(k = 1,2,\dots)$ are the eigenvalues resp. eigenfunctions of the operator $-\Delta$ with zero boundary conditions. We write formally the Fourier series of the solution $u(x,t)$ and the perturbation $\xi(x,t)$ to the (linear) equation (5.15) as

$$u(x,t) = \sum_{k=1}^{\infty} u^k(t)e_k \quad \text{and} \quad \xi(x,t) = \sum_{k=1}^{\infty}\xi^k(t)e_k\,. \tag{5.16}$$

If we introduce the Fourier transforms $\tilde{u}$ and $\tilde{\xi}$ of (5.16) with respect to the time variable we get from (5.15) for $k = 1,2,\dots$ the equations

$$-\omega^2\tilde{u}^k(i\omega) + 2i\omega\varepsilon\tilde{u}^k(i\omega) + \lambda_k\tilde{u}^k(i\omega) = -\alpha\sqrt{\lambda_k}\,\tilde{\xi}^k(i\omega)\,. \tag{5.17}$$

It follows from (5.17) that for $k = 1,2,\dots$

$$\tilde{u}^k = \chi(i\omega,\lambda_k)\,\tilde{\xi}^k\,, \tag{5.18}$$

where

$$\chi(i\omega,\lambda_k) = \left(-\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k\right)^{-1}(\alpha\lambda_k)\,, \quad \forall\omega\in\mathbb{R} : -\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k \neq 0\,. \tag{5.19}$$

In order to check the sufficient conditions for Theorem 4.2 we consider the functional

$$J(w,\xi) := \mathrm{Re}\int_0^\infty\int_0^l \left(\mu|w|^2 - w\xi^*\right)dx\,dt\,. \tag{5.20}$$

Using the Parseval equality for (5.20) with

$$|\tilde{w}|^2 = \sum_{k=1}^{\infty} \lambda_k |\tilde{u}^k|^2 = \sum_{k=1}^{\infty} \lambda_k |\tilde{u}^k|^2 = \sum_{k=1}^{\infty} \lambda_k |\chi(i\omega, \lambda_k)|^2 |\tilde{\xi}^k|^2$$

and

$$\tilde{w}\,\tilde{\xi}^* = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\tilde{u}^k\,(\tilde{\xi}^k)^* = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\chi\,(i\omega, \lambda_k)|\tilde{\xi}^k|^2\,,$$

we conclude ([1, 21]) that the functional (5.20) is bounded from above if and only if the functional

$$\operatorname{Re} \int_{-\infty}^{+\infty} \int_0^l [\mu \left(\sum_{k=1}^{\infty} \lambda_k |\chi(i\omega, \lambda_k)|^2 |\tilde{\xi}^k|^2 - \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\chi\,(i\omega, \lambda_k)|\tilde{\xi}^k|^2 \right] dx d\omega \tag{5.21}$$

is bounded on the subspace of Fourier-transforms defined by (5.18), (5.19) or, using again a result of Arov and Yakubovich ([1]), that the frequency-domain condition

$$\begin{aligned} &\mu\lambda_k |\chi\,(i\omega, \lambda_k)|^2 - \sqrt{\lambda_k}\operatorname{Re}\chi(i\omega, \lambda_k) < 0\,, \\ &\forall \omega \in \mathbb{R} : -\omega^2 + 2\,i\omega\varepsilon + \alpha\lambda_k \neq 0\,,\ k = 1, 2, \dots\,, \end{aligned} \tag{5.22}$$

is satisfied, where $\chi\,(i\omega, \lambda_k) = (-\omega^2 + 2i\omega\varepsilon + \alpha\lambda_k)^{-1}(-\alpha\sqrt{\lambda_k})$. Clearly, (5.22) describes a certain domain Q in the space of parameters $\mu > 0, \varepsilon > 0, \alpha \in \mathbb{R}$. Theorem 4.2 shows that (5.14), associated with (5.15),(5.2),(5.3) is absolutely stable with respect to the observation $z = (y_1, y_2)$, if the parameter from Q also guarantee the minimal stability of (5.14). □

6 Absolute observation convergence of evolutionary inequalities

In this section, we consider the observation stability of the difference of two arbitrary solutions of inequality (2.1), (2.2), i.e., convergence properties with respect to the observation (2.3) of this inequality, in order to get some information about the convergence of arbitrary solutions to the stationary set.
Let us start with the definition of a new class of nonlinearities for the case $\Xi = W$.

Definition 6.1 *Suppose in (2.1), (2.2) that $\Xi = W$ and assume that $\mu_0 > 0$ is an arbitrary number. The class $\mathcal{N}^c(\mu_0)$ consists of all maps $\varphi : \mathbb{R}_+ \times W \to W$ such that*

$$\begin{aligned} &0 \le (\xi_1 - \xi_2\,,\, w_1 - w_2)_W \le \mu_0 \|w_1 - w_2\|_W^2\,, \\ &\forall\, t \ge 0\,, \quad \forall\, w_1, w_2 \in W\,, \quad \forall\, \xi_1 \in \varphi(t, w_1)\,, \quad \forall\, \xi_2 \in \varphi(t, w_2)\,. \end{aligned} \tag{6.1}$$

Now we are in the position to give a precise interpretation of observation convergence (see also [13, 17, 21]).

Definition 6.2 *The inequality (2.1), (2.2) is said to be* ***absolutely convergent*** *(in the classes $\mathcal{N}^c(\mu_0)$ and $\mathcal{M}(d)$)* ***with respect to the observation*** *z from (2.3) if there exist two constants $c > 0$ and $c_2 > 0$, which depend only on $A, B, \mathcal{N}^c(\mu_0)$ and $\mathcal{M}(d)$, such that for any two solutions $\{y(\cdot), \xi(\cdot)\}$ and $\{\bar{y}(\cdot), \bar{\xi}(\cdot)\}$ of (2.1), (2.2) we have*

$$\|D\,(y(\cdot) - \bar{y}(\cdot)) + E\,(\xi(\cdot) - \bar{\xi}(\cdot))\|^2_{2,Z} \le c_1(\|y_0 - \bar{y}_0\|_0^2 + c_2)\,. \tag{6.2}$$

Theorem 6.1 *Suppose that the operator A from (2.1) generates a C_0-group on $Y_0, B \in \mathcal{L}\,(\Xi, Y_0)$, the pairs (A^c, B^c) and $(-A^c, B^c)$ are L^2-controllable, and (2.1), (2.2) with $f \equiv 0$ and $\Xi = W$ is minimally stable, i.e., there exists an operator $K \in \mathcal{L}\,(Y_1, W)$ such that $A + BK$ is stable, $(Kv, Cv)_W - \frac{1}{\mu_0}\|Kv\|^2_W \ge 0\,, \quad \forall\, v \in Y_1\,,$ and the pair $(A+BK, D+EK)$ is observable. Suppose also that $\varphi \in \mathcal{N}^c(\mu_0)$ and there exists an $\varepsilon > 0$ such that*

$$\begin{aligned}\mathrm{Re}\,(\xi, \chi^{(w)}(i\omega)\xi)_{W^c} - \tfrac{1}{\mu_0}\|\xi\|^2_{W^c} &\le -\varepsilon\|\chi^{(z)}(i\omega)\xi\|^2_{W^c}\,,\\ &\forall\,\omega \in \mathbb{R} : i\omega \notin \sigma(A^c)\,,\ \forall\,\xi \in W^c\,.\end{aligned} \tag{6.3}$$

Then inequality (2.1), (2.2) is absolutely convergent in the classes $\mathcal{N}^c(\mu_0)$ and $\mathcal{M}(d)$ with respect to the observation z.

Proof Suppose that $\{y(\cdot), \xi(\cdot)\}$ and $\{\bar{y}(\cdot), \bar{\xi}(\cdot)\}$ are two arbitrary solutions of (2.1), (2.2). If we consider in (2.1) the solution $\{y(\cdot), \xi(\cdot)\}$ and take the test function $\eta = \bar{y}$ we get

$$(\dot{y} - Ay - B\xi - f(t), \bar{y} - y)_{-1,1} + \psi\,(\bar{y}) - \psi\,(y) \ge 0\,. \tag{6.4}$$

Now we consider (2.1) with the solution $\{\bar{y}(\cdot), \bar{\xi}(\cdot)\}$ and take the test function $\eta = y$ in order to get

$$(\dot{\bar{y}} - A\bar{y} - B\bar{\xi} - f(t), y - \bar{y})_{-1,1} + \psi\,(y) - \psi\,(\bar{y}) \ge 0\,. \tag{6.5}$$

Suppose $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ is an arbitrary linear operator and write (6.4) with
$\bar{y} - y = P(\bar{y} - y)$ and (6.5) with $y - \bar{y} = P(y - \bar{y})$, i.e.,

$$(\dot{y} - Ay - B\xi - f(t)\,,\ P(\bar{y} - y))_{-1,1} + \psi\,(y - P(\bar{y} - y)) - \psi\,(y) \ge 0 \tag{6.6}$$

and

$$(\dot{\bar{y}} - A\bar{y} - B\bar{\xi} - f(t),\ P\,(y - \bar{y}))_{-1,1} + \psi\,(\bar{y} + P\,(y - \bar{y})) - \psi(\bar{y}) \ge 0\,. \tag{6.7}$$

If we take the sum of (6.6) and (6.7) we get the inequality

$$\begin{aligned}&(\dot{y} - \dot{\bar{y}}\,, P\,(y - \bar{y}))_{-1,1} - (A(y - \bar{y}) + B(\xi - \bar{\xi})\,, P\,(y - \bar{y}))_{-1,1}\\ &-\psi\,(y - P\,(\bar{y} - y)) + \psi(y) - \psi(\bar{y} + P\,(y - \bar{y})) + \psi(\bar{y}) \le 0\,.\end{aligned} \tag{6.8}$$

Recall now that the frequency-domain inequality (6.3) together with the other assumptions of Theorem 6.1 guarantee on the base of Theorem 3.1b) the existence of a (real) operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and a number $\delta > 0$ such that

$$(Av + B\zeta\,, P\,v)_{-1,1} + (\zeta, Cv)_W - \tfrac{1}{\mu_0}\|\zeta\|_W^2 \le -\delta\|Dv + E\,\zeta\|_Z^2\,, \qquad \forall\, v \in Y_1,\ \forall \zeta \in W\,. \tag{6.9}$$

Using the fact that (2.1), (2.2) is minimally stable with some $K \in \mathcal{L}(Y_1, W)$ we conclude from (6.9) that $((A+BK)v, Pv)_{-1,1} \le -\delta\|(D+KE)v\|_Z^2\,, \quad \forall v \in Y_1$. From this inequality it follows that P is non-negative.
For the considered arbitrary solutions $\{y(\cdot), \xi\,(\cdot)\}$ and $\{\bar y(\cdot), \bar\xi(\cdot)\}$ of (2.1), (2.2) the inequalities (6.8) and (6.9) imply that

$$\begin{aligned}&(\dot y - \dot{\bar y}\,, P\,(y - \bar y))_{-1,1} + (\xi - \bar\xi, C(y - \bar y))_W - \tfrac{1}{\mu_0}\|\xi - \bar\xi\|_W^2\\ &+ \delta\|D\,(y - \bar y) + E\,(\xi - \bar\xi)\|_Z^2 \le 0\,.\end{aligned} \tag{6.10}$$

The integration of (6.10) on an arbitrary time interval $[0, t]$ $(t > 0)$ gives for the Lyapunov-functional $V(y) := (y, Py)_0$ on the base of (6.1) and the fact that $\psi \in \mathcal{M}(d)$ the inequality

$$V(y(t) - \bar y(t)) - V(y(0) - \bar y(0)) + \delta \int_0^t \|D(y(\tau) - \bar y\,(\tau)) + E(\xi(t) - \bar\xi(\tau))\|_Z^2\, d\tau \le c\,, \tag{6.11}$$

where c is a sufficiently large constant. Since $V(y(t) - \bar y(t)) \ge 0$ the inequality (6.11) implies

$$\delta \int_0^\infty \|D(y(\tau) - \bar y\,(\tau)) + E\,(\xi(\tau) - \bar\xi(\tau))\|_Z^2\, d\tau \le V(y(0) - \bar y(0)) + c\,.$$

■

In the sequel we investigate the special situation of the convergence of an arbitrary solution of (2.1), (2.2) to a stationary solution. In case when such a stationary solution is unique we can use Theorem 6.1. In many applications, however, the inequality (2.1), (2.2) has a continuum of stationary solutions and the approach of Theorem 6.1 is not applicable.
Let us consider the autonomous inequality (2.1), (2.2) with $\varphi(t, w) \equiv \varphi(w)$ and $f(t) \equiv 0$, i.e.,

$$(\dot y - Ay - B\xi, \eta - y)_{-1,1} + \psi(\eta) - \psi(y) \ge 0\,, \quad \forall y \in Y_1\,, \tag{6.12}$$
$$y\,(0) = y_0 \in Y_0\,,$$
$$w\,(t) = Cy(t)\,,\ \xi(t) \in \varphi(w(t))\,,\ \xi(0) = \xi_0 \in \mathcal{E}\,(y_0)\,, \tag{6.13}$$
$$z\,(t) = Dy(t) + E\,\xi(t)\,. \tag{6.14}$$

Our aim is to show that under certain observation conditions any solution of (6.12), (6.13) converges to the stationary set of the inequality. Let us start with some definitions ([13, 17, 21]).

Definition 6.3 *Any constant solution $\{y(\cdot), \xi(\cdot)\}$ of (6.12), (6.13), i.e., $y(t) \equiv \text{const}$, $\xi(t) \equiv \text{const}$ on $\mathbb{R}_+$, is called a* ***stationary solution*** *of (6.12), (6.13) . The set of all stationary solutions of (6.12), (6.13) is called the* ***stationary set*** *of (6.12), (6.13) and denoted by $L \times \Lambda \subset Y_1 \times \Xi$. We say that the solution $\{y(\cdot), \xi(\cdot)\}$ of (6.12), (6.13) is* ***weakly quasi-convergent*** *if $y(t) \to L$ in the weak sense as $t \to +\infty$. The inequality (6.12), (6.13) is called* ***weakly quasi-gradient-like*** *if every its solution is weakly quasi-convergent.*

It is clear from the definition that a pair $\{\bar{y}, \bar{\xi}\}$ is a stationary solution of (6.12), (6.13) if and only if

$$(-A\bar{y} - B\bar{\xi}\,,\, \eta - \bar{y})_{-1,1} + \psi(\eta) - \psi(\bar{y}) \geq 0\,, \quad \forall y \in Y_1\,, \tag{6.15}$$

$$\bar{y} \in Y_0\,,$$

$$\bar{w} = C\bar{y}\,,\ \bar{\xi} \in \varphi(\bar{w}) \cap \mathcal{E}(\bar{y})\,. \tag{6.16}$$

With respect to the stationary set described by (6.15), (6.16) we make the following assumptions.

(S1) The stationary set $L \times \Lambda$ of (6.15), (6.16) is non-empty and consists of isolated points only.

Remark 6.1 Under standard conditions the assumption **(S1)** is satisfied ([11, 24, 26]). If (6.15) can be written as

$$a\,(\bar{y}, \eta - \bar{y}) + c\,(\bar{y}, \eta - \bar{y})_{-1,1} + \psi(\eta) - \psi(\bar{y}) \geq 0\,, \quad \forall y \in Y_1\,, \tag{6.17}$$

where $a(y,\eta) = a\,(\eta, y)$ is a quadratic form on $Y_1 \times Y_1$ and $c \in \mathbb{R}$ is a number, the problem (6.15) is equivalent to the minimisation of the functional

$$J(\eta) := \frac{1}{2}\,\left[a\,(\eta,\eta) + c\,\|\eta\|_1^2\right] + \psi(\eta)$$

on Y_1 . □

Let us formulate some simplifications of system (6.12), (6.13) . Assume that

(S2) $\qquad \psi \equiv 0 \quad \text{and} \quad \Xi = W = \mathbb{R}\,.$

It follows that $C \in \mathcal{L}(Y_{-1}, \mathbb{R})$, $B \in \mathcal{L}(\mathbb{R}, Y_{-1})$, $D \in \mathcal{L}(Y_1, \mathbb{R})$, $E \in \mathcal{L}(\mathbb{R}, \mathbb{R})$, and $\varphi : \mathbb{R} \to 2^{\mathbb{R}}$. Let us further assume, as in the case of ODE's with retarded arguments, ([21]) that φ is piecewise continuous, discontinuous in $w = 0$ and with the set of isolated discontinuity points $\{w_j\}$. For any discontinuity point w_j the set $\varphi(w_j) \subset \mathbb{R}$ is assumed as a closed interval such that $\varphi(w_j) \supset [\liminf_{w \to w_j} \varphi(w)\,, \limsup_{w \to w_j} \varphi(w)]$. Then ([13, 16]) φ is an upper-semicontinuous function. The local solution in the sense of Sect. 2 is assumed to exist globally. In order to construct a quadratic constraint F for φ we assume that φ satisfies the following property.

(S3) $$\xi\, w \geq 0\,, \qquad \forall\, w \in \mathbb{R}, \qquad \forall\, \xi \in \varphi(w)\,. \tag{6.18}$$

The last condition implies that $\varphi(0) = [\varphi_1, \varphi_2]$, with $\varphi_1 \le 0$ and $\varphi_2 \ge 0$. Assume $\varphi_1 < 0$ and $\varphi_2 > 0$.
In mechanics many systems with dry friction are described by equation (6.12), (6.13) with nonlinearities satisfying **(S1) - (S3)**. Typical solutions of such systems are sliding solutions. Let us give the formal definition which goes back to [13, 21, 27].

Definition 6.4 *A solution $\{y(\cdot), \xi(\cdot)\}$ of (6.12), (6.13) is called* ***sliding solution*** *on the time interval (t_1, t_2) if $w(t) = Cy(t) \equiv w_j$ on (t_1, t_2), where w_j is a discontinuity point of φ.*

Let us investigate observation properties of sliding solutions associated with the discontinuity $w = 0$ of φ. Consider for $s \in \rho(A^c)$ the transfer operator $\chi^{(w)}(s) = C^c(sI^c - A^c)^{-1}B^c = C^c\chi(s)$ and assume that $0 \in \rho(A^c)$ and $\chi^{(w)}(0) = 0$. It follows from (6.12), (6.13) that the stationary sliding solutions of the system corresponding to the discontinuity $w = 0$ of φ have the form $y(t) \equiv \bar{y}, \xi(t) = \bar{\xi}$, where $\bar{\xi} \in \varphi(0)$, $\bar{y} = \xi A^{-1}B$.
From Theorem 4.1 we derive now sufficient conditions for absolute observation stability. Note, that for the case of ODE's with retarded arguments similar conditions are derived in [21].

Theorem 6.2 *Suppose that for the linear part of (6.12), (6.13) the assumptions of Theorem 6.1 are satisfied, (6.12), (6.13) is minimally stable, $\chi^{(w)}(0) = 0$ and there exist parameters $\Theta_j > 0$ $(j = 1, 2)$ such that the following inequalities hold:*

$$\mathrm{Re}\left[(\Theta_1 + i\omega\,\Theta_2)\,\chi^{(w)}(i\omega)\right] < 0\,, \quad \forall\,\omega \in \mathbb{R}\backslash\{0\}\,, \tag{6.19}$$

$$\lim_{\omega\to 0} \frac{1}{\omega^2}\,\mathrm{Re}\left[(\Theta_1 + i\omega\,\Theta_2)\,\chi^{(w)}(i\omega)\right] < 0\,, \tag{6.20}$$

and

$$\liminf_{|\omega|\to+\infty} \mathrm{Re}\left[(\Theta_1 + i\omega\Theta_2)\,\chi^{(w)}(i\omega)\right] \le \Pi < 0\,. \tag{6.21}$$

Then (6.12), (6.13) is absolutely stable with respect to all observations $z_1(t) = Dy(t)$ and $z_2(t) = D\dot{y}(t) = DAy(t) + DB\xi(t)$, where $D \in \mathcal{L}(Y_1, \mathbb{R})$ is such that

$$DA^{-1}B = 0\,. \tag{6.22}$$

Proof We consider the quadratic form

$$F + G = \Theta_1 F^1 + \Theta_2 G^1 := \Theta_1 Cy\,\xi + \Theta_2 C\,[Ay + B\xi]\xi\,. \tag{6.23}$$

The first part $\Theta_1 F^1$ comes from the inequality (6.8), the second part $\Theta_2 G^1$ is introduced through (2.5). For $\xi = 0$ we have $F(y,0) + G(y,0) = 0$ and the system is minimally stable. Extend now $F + G$ to a Hermitian form $F^c + G^c$ on $\mathbb{C} \times \mathbb{C}$ by

$$F^c(y,\xi) + G^c(y,\xi) = \mathrm{Re}\,[\Theta_1 Cy\xi^* + \Theta_2 C(Ay + B\xi)\xi^*]\,. \tag{6.24}$$

If we consider the observations $z_1(t) = Dy(t)$ together with the quadratic form (6.23) the frequency-domain condition (3.6) becomes

$$\exists\,\delta > 0 : \Pi(i\omega) \le -\delta|D\chi(i\omega)|^2\,, \qquad \forall\omega \in \mathbb{R}, \tag{6.25}$$

with $\Pi(i\omega) := \mathrm{Re}\,[(\Theta_1 + i\omega\Theta_2)\chi^{(w)}(i\omega)]$. In [21] it is shown that (6.25) is true if (6.19)−(6.21) and (6.22) are satisfied. Thus the system (6.12), (6.13) is absolutely stable with respect to all observations $z_1(t) = Dy(t)$ with D satisfying (6.22).
If we consider now the observations $z_2(t) = DAy(t) + DB\xi(t)$ together with the quadratic form (6.23) the frequency-domain condition (3.6) becomes

$$\exists\,\delta > 0 : \Pi(i\omega) \le -\delta|i\omega D\chi(i\omega)|^2, \qquad \forall\omega \in \mathbb{R}. \tag{6.26}$$

It is easy to check (see [21]) that (6.26) is also satisfied provided that (6.19)−(6.21) and (6.22) are true. This shows that the system (6.12), (6.13) is also absolutely stable with respect to the observation z_2 ■

From Theorem 6.2 we get immediately the following.

Corollary 6.1 *Suppose that the assumptions of Theorem 6.1 are satisfied. Then every solution* $\{y(\cdot), \xi(\cdot)\}$ *of (6.12), (6.13) converges weakly to the set*

$$L := \{A^{-1}B\xi\ ,\ \xi \in \mathbb{R}\}\ ,$$

and, consequently, the system (6.12), (6.13) is weakly quasi-gradient like with respect to the stationary sliding solutions.

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